

# Fast Processing for the Short Time DFT Hilbert Transformer

## Short Time DFT Hilbert変換器のための 高速化処理

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**ABSTRACT** *An exact realization of the Hilbert transformer has been previously discussed with employing new concept of instantaneous spectrum on the basis of frequency domain Hilbert transform. The FFT processing structure yields some advances to the short time DFT Hilbert transformer to overcome the absolute defect by reducing the great processing amount without any loss of generality via employing interpolation of multi-rate sampling.*

*This novel transformer is also discussed to prevent the functional precision from synchronization error which occurs either between analyzer and synthesizer within the novel transformer or between two novel transformers installed at sending and receiving sites in radio communication systems.*

### 1. INTRODUCTION

It is as well known as important to reduce the spectrum occupancy and to prevent radio resource from exhausting by rapid popularization in radio communications. The Hilbert transformer used in SSB or RZ SSB modulator provides with indispensable function for eliminating one sideband from output signals to efficiently reduce occupied spectrum over radio channels (1).

Therefore, many investigations are keenly studied on realizing the Hilbert transformers (2). Especially, such transformer as shifting the phase of input signals on the frequency domain is eager to develop for the preciseness in functions(3). Unfortunately, spectrum exudate at frame edges introduces the transformer process-

ing distortion owing the existing DFT to low time resolution of averaging the spectrum a frame duration.

By employing instantaneous spectrum concept, a novel Hilbert transformer named by *Short Time DFT* (abbreviated as ST-DFT) *Hilbert Transformer* is successfully realized over the frequency domain without any distortion both in amplitude and phase-shifting(4). The ST-DFT transformer is examined to be error free and almost equal both in processing amount and functions to the existing standard Hilbert transformers of the minimax through computer simulations(5).

A circuitry configuration of the ST-DFT Hilbert transformer is categorized into three major blocks, namely, I :instantaneous spectrum analyzer, II :frequency domain Hilbert transformer, III :output signal synthesizer. The ST-DFT transformer is discussed to prevent its functional precision from synchronization error

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which occurs either between analyzer and synthesizer within the noble transformer or between two transformers installed at sending and receiving sites in radio communication systems.

Fast processing for the ST-DFT Hilbert transformers is introduced to overcome the defect of great processing amount with employing interpolation of the multi-rate sampling.

## 2. CONFIGURATION AND CHARACTERISTICS OF THE SHORT TIME DFT HILBERT TRANSFORMER

### 2.1 Processing Outline

The short time DFT Hilbert transformer features in shifting the phase of input signals on the frequency domain. All of input signals are at first analyzed into the instantaneous spectrum  $\Phi(n)$  in the ST-DFT Hilbert transformer as follows,

$$\Phi(n) = \{\phi_0(n) \ \phi_1(n) \ \phi_2(n) \ \cdots \ \phi_{N-1}(n)\}^T. \quad (1)$$

where,  $\phi_k(n)$  is a spectrum component at frequency index  $k$  of  $\Phi(n)$  at sampling clock  $n$ . The spectrum component  $\phi_k(n)$  is defined by short time DFT as,

$$\phi_k(n) = \sum_{r=-\infty}^{\infty} x(r)h(n-r)W_N^{-rk}, \quad (2)$$

here,  $W_N^{-rk} = \exp\{-j(2\pi rk/N)\}$ ,  
integer  $k$  is  $0 \leq k < N$ ,

$x(r)$  is an input data at sampling time  $r$ ,  
 $W_N^{-rk}$  is the same to the operator of existing DFT,  
 $h(*)$  is a significant window function defined by

$$h(p) = \begin{cases} 1, & \text{if } p = 0. \\ 0, & \text{if } p = 2Nu, \end{cases} \quad (3)$$

$u$  is non zero integer.

An  $N$  frame length Nyquist window function truncated with  $2m$  frame number  $h(p)$

$$h(p) = \frac{\sin(p\pi/N)}{p\pi/N}, \quad -mN \leq p \leq mN, \quad (4)$$

may be employed as the significant window function. More sophisticated window will be offered by the same group of this author under detailed consideration for prototype filter in the decimation.

At second, the Hilbert transform is performed on the frequency domain by exchanging real and imaginary part of each component of  $\Phi(n)$  with each other to yield transformed instantaneous spectrum  $\widehat{\Phi}(n)$ . The  $\widehat{\Phi}(n)$  should hold restrict conditions for physical existence of consisting of complex conjugate components with symmetric axis at index  $N/2$ . Here,  $N$  means the inner frame sampled data number.

Output signals  $\widehat{y}(n)$  are finally produced from the transformed spectrum  $\widehat{\Phi}(n)$  through short time IFT synthesizer as follows.

$$\begin{aligned} \widehat{y}(n) &= \frac{1}{N} \sum_{k=1}^{N/2-1} \left\{ \widehat{\phi}_k(n)W_N^{nk} + \widehat{\phi}_{N-k}(n)W_N^{n(N-k)} \right\} \\ &= \frac{2}{N} \sum_{k=1}^{N/2-1} \text{Real} \left\{ \widehat{\phi}_k(n)W_N^{nk} \right\} \end{aligned} \quad (5)$$

These processing steps of instantaneous spectrum analysis and phase shifting being combined into single operation, the frequency domain Hilbert transform operator  $\widehat{W}_N^{-rk}$  is consequently given as follows,

$$\widehat{W}_N^{-rk} = \begin{cases} \exp\{-j(2\pi rk/N + \pi/2)\}, & \text{if } 0 < k < N/2 \\ 0, & \text{if } k = 0, N/2 \\ \exp\{-j(2\pi rk/N - \pi/2)\}, & \text{if } N/2 < k < N \end{cases} \quad (6)$$

here,  $j$  is complex unit,  $j = \sqrt{-1}$ .

### 2.2 Circuitry Configuration and Unit Sample Response

The ST-DFT Hilbert transformer consists of three major blocks as shown in fig.1. The first block is the ST-DFT analyzer and consists of

$N/2 - 1$  modules in which every component  $\phi_k(n)$  is yielded. Inner product of  $x(n)$  and  $W_N^{-rk}$  in eq.2 is performed of modulating the input  $x(n)$  with complex carrier  $W_N^{-rk}$ . Convolution  $\{x(r)W_N^{-rk}\}$  and  $h(r)$  in the same equation is also interpreted as low - pass filtering the modulated signal  $\{x(r)W_N^{-rk}\}$  by  $h(r)$ .

The second block is a Hilbert transformer on the frequency domain. This block is dominant in function, however, it is so simply implemented as two crossing wires to exchange the real with the imaginary part of  $\phi_k(n)$ . The first and second blocks are practically combined together to get  $\widehat{\phi}_k(n)$  directly in frequency index wise by adopting  $\widehat{W}_N^{-rk}$  instead of  $W_N^{-rk}$  during the modulation.

The last is a ST - IFT synthesizer to produce time domain Hilbert transformed signals. In similar to the first block, ST - IFT synthesizer is performed of modulating Hilbert transformed spectrum component  $\widehat{\phi}_k(n)$  with complex carrier  $W_N^{nk}$ . The unit sample response  $Is(n)$  of the ST - DFT Hilbert transformer is given by eq.7.

$$Is(n) = \begin{cases} \frac{2 \sin(2\pi n/N) \cdot \sin(\pi n/N)}{N \{1 - \cos(2\pi n/N)\} \pi n/N} \\ = 2 \frac{\cos(\pi n/N)}{\pi n}, & \text{if } n \text{ is odd.} \\ 0, & \text{if } n \text{ is even.} \end{cases} \quad (7)$$

The unit sample response  $Im(n)$  of the Rabiner'

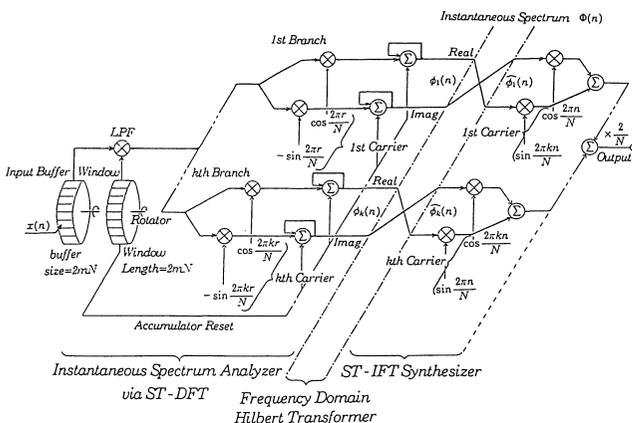


Fig. 1 Configuration of short time DFT Hilbert transformers.

s minimax FIR Hilbert (ab. in minimax) transformer is given by eq.8.

$$Im(n) = \frac{2 \sin^2(\pi n/2)}{\pi n} = \frac{1 - \cos(\pi n)}{\pi n} = \begin{cases} \frac{2}{\pi n}, & \text{if } n \text{ is odd.} \\ 0, & \text{if } n \text{ is even.} \end{cases} \quad (8)$$

It is shown in both eqs.7 and 8 that the ST - DFT Hilbert transformer enhances the minimax transformer. That is,

$$\lim_{N \rightarrow \infty} Is(n) = \lim_{N \rightarrow \infty} \frac{2 \cos(\pi n/N)}{\pi n} = \frac{2}{\pi n} = Im(n) \quad (9)$$

Phase shifting error both of the ST - DFT and minimax transformers are so accurate as detecting no error by  $10^{-9}$  degree scale as shown in fig.2(a). Fig.2 also shows that there exists no difference between power spectrums of ST - DFT and minimax Hilbert transformers, only except neighborings around 0 and  $\pi$  radian. Amplitude error of the ST - DFT transformer is shown not to exceed that of the minimax in absolute over all frequency domain.

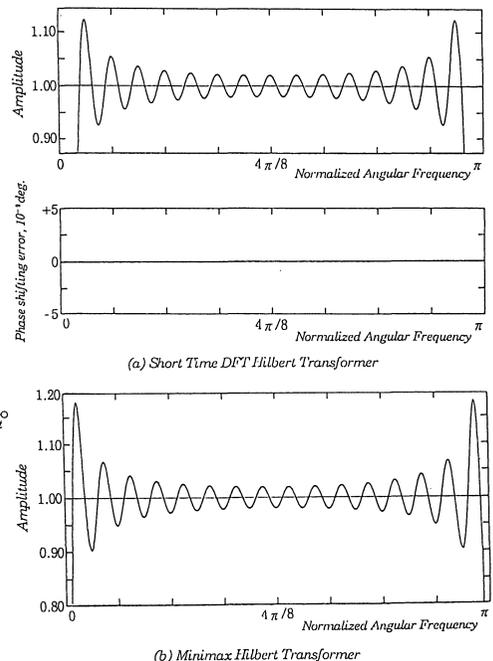


Fig. 2 Comparison of system functions between ST - DFT and minimax Hilbert transformers.

### 3. ROBUSTNESS IN SYNCHRONIZATION ERROR

#### 3.1 Robustness in Inner - Transformer Synchronization Error

It is already shown that ST - DFT Hilbert transformers consist of ST - DFT analyzer and ST - IFT synthesizer. However, it is difficult to coincide with both time bases in these modules according to processing property. It introduce excessive processing delay to coincide time bases with each other.

Let's consider what effect happens to the output signals with giving arbitrary delay  $\delta$  between analyzer and synthesizer as shown in fig.3. Here, the time base of analyzer is taken to be standard of processing in the transformer. The transformed signals  $\tilde{y}(n)$  are given by,

$$\tilde{y}(n) = \frac{1}{N} \sum_{k=0}^{N-1} \hat{\phi}_k(n) W_N^{(n-\delta)k} = \frac{1}{N} \sum_{k=0}^{N-1} \hat{\phi}_k(n) W_N^{nk} W_N^{-\delta k} (= \hat{y}(n) Z^{-\delta}). \tag{10}$$

Factor  $W_N^{-\delta k}$  in eq.10 is time invariant and linear with the frequency. That is, this factor  $W_N^{-\delta k}$  is shown as well known to be an operator which gives delay  $\delta$  to  $\hat{y}(n)$  of synchronization error free output signals. Where Z transform is employed, the output signals are given by bracketed term in eq.10. ST - DFT Hilbert transformer is consequently shown to hold robustness in synchronization error between ST - DFT analyzer and synthesizer in itself.

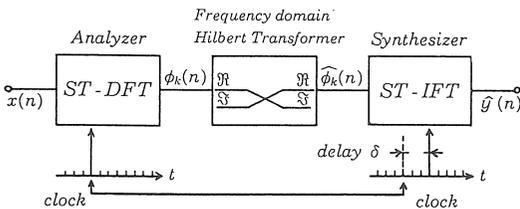


Fig. 3 Scheme of inner transformer synchronization error,  $\delta$ .

#### 3.2 Robustness in Intra - Transformer Synchronization Error

In general, Hilbert transformers are employed at both source and destination sites in communication systems. It is ease to understand that synchronizing processing time base among communication sites is so difficult as becoming to a big problem in ISDN. Especially, synchronization is seemed to be impossible in radio communication systems. Therefore, we should discuss about intra - transformer synchronization error.

Let's consider instantaneous spectrum  $\tilde{\phi}_k(n)$  at one site with time delay  $\epsilon$ , where exists time delay  $\epsilon$  between input signals as shown in fig.4.

$$\tilde{\phi}_k(n) = \sum_{r=-\infty}^{\infty} x(r - \epsilon) h(n - r) W_N^{-rk} \tag{11}$$

Instantaneous spectrum  $\tilde{\phi}_k(n + \epsilon)$  at time  $n + \epsilon$  is deduced from eq.11 as follows.

$$\begin{aligned} \tilde{\phi}_k(n + \epsilon) &= \sum_{r=-\infty}^{\infty} x(r - \epsilon) h(n + \epsilon - r) W_N^{-rk} W_N^{+\epsilon k - \epsilon k} \\ &= \sum_{r=-\infty}^{\infty} x(r - \epsilon) h\{n - (r - \epsilon)\} W_N^{-(r-\epsilon)k} W_N^{-\epsilon k} \end{aligned}$$

Here, set  $s$  to be  $r - \epsilon$ ,  $\tilde{\phi}_k(n + \epsilon)$  is given by eq.12.

$$\begin{aligned} \tilde{\phi}_k(n + \epsilon) &= \sum_{s=-\infty}^{\infty} x(s) h(n - s) W_N^{-sk} W_N^{-\epsilon k} \\ &= \phi_k(n) z^{-\epsilon} \end{aligned} \tag{12}$$

The instantaneous spectrum  $\tilde{\phi}_k(n + \epsilon)$  while receiving signals are delayed by  $\epsilon$  is shown to coincide with the instantaneous spectrum  $\phi_k(n)$  only except of delay  $\epsilon$ .

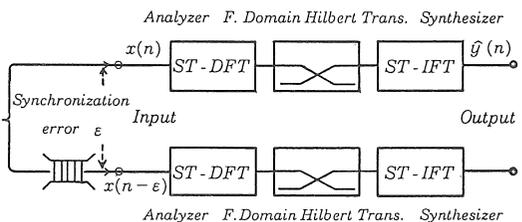


Fig. 4 Scheme of intra transformers synchronization error,  $\epsilon$ .

#### 4. FAST PROCESSING FOR ST-DFT HILBERT TRANSFORMER

Literal processing based on eqs.2, 5 and 6 requires a great deal of computing power through ST-DFT Hilbert transform. The Hilbert transformed output signals  $\widehat{y}(n)$  are synthesized as shown in fig.5 from interpolated instantaneous spectrum  $\widehat{\Phi}(n)$ , whose components  $\widehat{\phi}_k(n)$  are reproduced from  $\widehat{\phi}_k(r)$  at every  $R$  sampling as follows(6).

$$\widehat{y}(n) = \sum_{k=0}^{N-1} \frac{1}{N} \left\{ \sum_{r=L^-}^{L^+} f(n-rR) \widehat{\phi}_k(r) \right\} W_N^{mk}, \quad (13)$$

Where,  $L^- = \left[ \frac{n}{R} \right] - Q + 1$ ,  $L^+ = \left[ \frac{n}{R} \right] + Q$ , (14)

here  $[A]$  represents the largest integer contained  $A$ ,

$\widehat{\phi}_k(r)$  means the decimated instantaneous spectrum by every  $R$  sampling periods,  $\widehat{\phi}_k(r) = \widehat{\phi}_k(rR)$ , and  $f(n-rR)$  is such an interpolation filter as Lagrange, given by

$$f(n-rR) = \frac{(-1)^{r+Q} \prod_{i=1}^Q \left( \frac{n}{R} + Q - i \right)}{(Q-1+r)! (Q-r)! \left( \frac{n}{R} - r \right)}. \quad (15)$$

As the summations are defined over finite terms both for  $k$  and  $r$ , eq.13 stands for interchanging the order of summations. Therefore,

$$\widehat{y}(n) = \sum_{r=L^-}^{L^+} f(n-rR) \left\{ \frac{1}{N} \sum_{k=0}^{N-1} \widehat{\phi}_k(r) W_N^{mk} \right\} \quad (16)$$

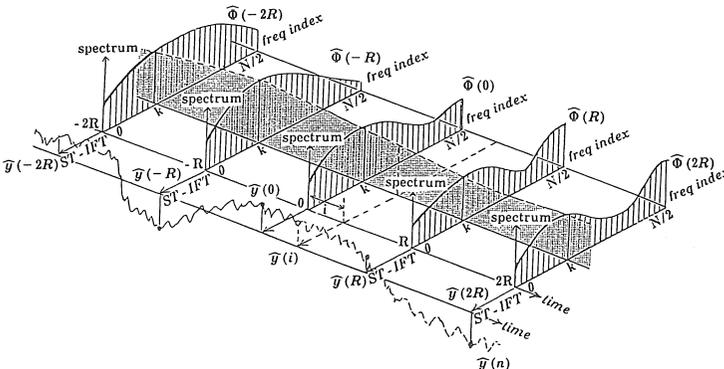


Fig. 5 Fast processing based on frequency domain interpolation.

So long as  $R \leq N$ , the Hilbert transformed output  $\widehat{y}(n)$  are precisely regenerated from the decimated instantaneous spectrums. The summation for  $k$  on right hand of eq.16 represents IFT. Output signals  $\widehat{y}(n)$  of the speeded up ST-DFT Hilbert transformer, which are hereafter called by "fast ST-DFT", are finally given by eq.17.

$$\widehat{y}(n) = \frac{1}{N} \sum_{r=L^-}^{L^+} f(n-rR) \widehat{s}_r(n), \quad (17)$$

where,  $\widehat{s}_r(n) = \frac{1}{N} \sum_{k=0}^{N-1} \widehat{\phi}_k(rR) W_N^{mk}$ .

#### 5. EXPERIMENTAL RESULTS

The fast ST-DFT Hilbert transformer is experimented with computer simulations to substantiate its facilities. Owing to employing interpolation to reduce the processing amount, the system function of the fast ST-DFT transformer becomes to vary with both the time of input unit sample and interpolating duration  $R$ . That is, while the unit sample is given at origin sampling points, i.e. input signal  $\delta(n-\tau)=0, \tau=0$ , the precision of the transformed output signals is, in regardless of value  $R$ , within  $10^{-9}$  degree in phase shifting error and within envelope of the minimax shown in fig.2(b) in amplitude error.

However, if  $\tau$  goes to non-zero number, 1 or  $2 \ll N$ , i.e.  $\delta(n-\tau)=0, \tau \neq 0$ , the frequency responses are remarkably degraded as shown in fig.6(a) when interpolation duration  $R$  being set

up to the maximum value  $N$ . Here  $N$  is 32. Amplitude error keeps peak values within the envelope of the minimax amplitude. Phase shifting error is scared away beyond  $\pm 4$  degree from the aimed -90 degree. Worst phase shifting error occurs in setting  $\tau = N/2 - 1$  up to 22.5 degree of  $\pi/4$ .

While interpolation duration  $R$  is set to 11, nearly to one third of  $N(=32)$ , amplitude frequency re

sponses keep errors within the minimax's envelopes, and the phase shifting error is improved within  $\pm 0.4$  degree of one tenth of  $R=N$ , as shown in fig.6(b). The maximum phase shifting error is observed at  $\tau=5$  or 6 of  $N/2 - 1$  within  $\pm 0.68$  degree, where the amplitude error is shrunk to 1% of one fifth of fig.6(b) around normalized frequency  $\pi/2$ .

## 6. CONCLUSION

A noble Hilbert transformer was discussed with

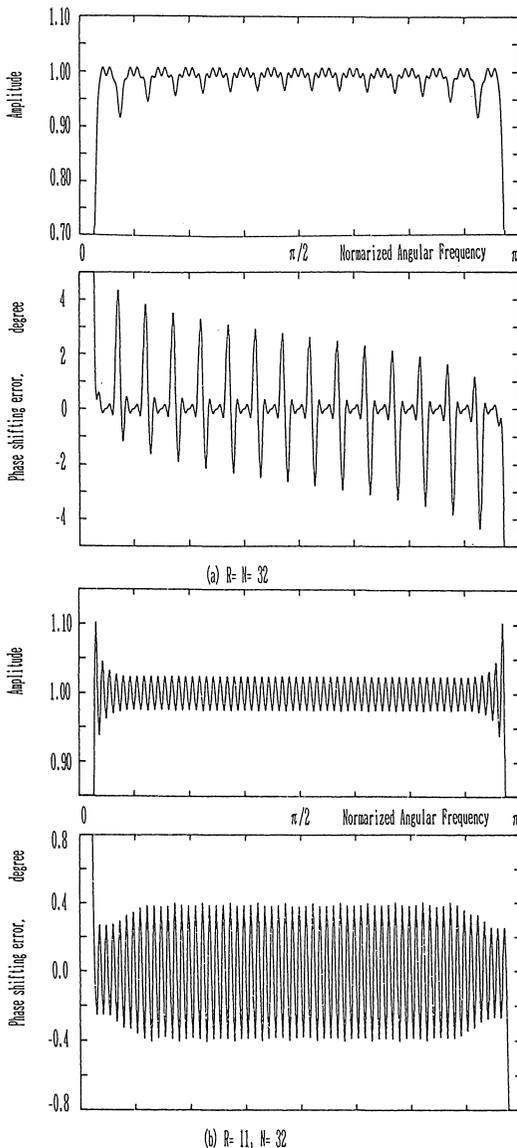


Fig. 6 Fast ST - DFT Hilbert transformer characteristics,  $\tau=2$ .

emphasis on the instantaneous spectrum signal processing, through its circuitry configuration, fast processing algorithm, and frequency responses. A primitive truncated Nyquist being employed as the significant window  $h(*)$ , ST-DFT Hilbert transformer can obtain preciseness equal to the existing Rabiner's pre-optimized minimax one in both phase shifting and rapidness of transient response. Farther studies will improve such primitive instantaneous spectrum signal processing as done in minimax Hilbert transformer by Remez algorithm.

Multi-rate sampling have been successfully introduced to reduce the processing amount of ST-DFT transformers without almost any distortion, where the interpolation duration  $R$  is set to nearly equal to one third of frame length  $N$ .

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