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〈査読付論文〉

Hopf algebra structures on the quantum affine superalgebras of type $D^{(1)}(2,1;x)$ $(x \in \mathbb{C} \setminus \{0,-1\})$

$D^{(1)}(2,1;x)$ $(x \in \mathbb{C} \setminus \{0,-1\})$ 型量子アフィン・スーパー代数上のホップ代数構造

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Abstract. We will construct Hopf algebra structures on the quantum affine superalgebras of type $D^{(1)}(2,1;x)$ $(x \in \mathbb{C} \setminus \{0,-1\})$.

1 Introduction

In [2], we prove the existence of the bilinear forms on the quantum affine superalgebras of type $D^{(1)}(2,1;x)$ by using a manner similar to Tanisaki's in [5], where $x \in \mathbb{C} \setminus \{0,-1\}$. Our purpose of this paper is to give the foundations of the paper [2]. Especially, Proposition 3.4 is an important key to prove the existence of the bilinear forms.

It should be remarked that Theorem 3.5 of this paper is nothing but Theorem 4.5(1) of the paper [1]. However, in the paper [1], they do not give the proof of Theorem 4.5(1), and they comment that Theorem 4.5(1) can be checked by using the computer algebra program Mathematica. In this paper, we will give a detailed proof of Theorem 3.5, i.e., the existence of Hopf algebra structures on the quantum affine superalgebras of type $D^{(1)}(2,1;x)$ by using Proposition 3.4.

This paper is organized as follows. In section 2, we recall the definition of the quantum affine superalgebras of type $D^{(1)}(2,1;x)$ and give several formulas. In section 3, we give several formulas related to the existence of Hopf algebra structures on the quantum affine superalgebras of type $D^{(1)}(2,1;x)$ (see Proposition 3.4), and then construct Hopf algebra structures on the quantum affine superalgebras of type $D^{(1)}(2,1;x)$. In section 4, we give detailed proofs of several equalities which are used to prove Proposition 3.4.

2 The quantum affine superalgebras of type $D^{(1)}(2,1;x)$

First of all, we would like to mention that the notations of this paper follow that of [1] and [2]. By referring to the papers, we will omit the detailed description of the notations.

In this section, we will give several formulas on the quantum affine superalgebra of type $D^{(1)}(2,1;x)$, where $x \in \mathbb{C} \setminus \{0,-1\}$. For each $d \in \mathcal{D}$, the associative algebra $\mathcal{U}'_d$ over $\mathbb{C}$ with the unit 1 is defined by the generators

$$\sigma_d,\ K_{i,d}^{\pm\frac{1}{2}},\ E_{i,d},\ F_{i,d} \quad (i \in \mathrm{I}),$$

and the following relations

$$XY = YX \quad \text{for} \quad X, Y \in \{\sigma_d, K_{i,d}^{\pm\frac{1}{2}}\}, \tag{2.1}$$

$$\sigma_d^2 = 1, \quad K_{i,d}^{\frac{1}{2}} K_{i,d}^{-\frac{1}{2}} = K_{i,d}^{-\frac{1}{2}} K_{i,d}^{\frac{1}{2}} = 1, \tag{2.2}$$

$$\sigma_d E_{i,d} \sigma_d = (-1)^{p(\alpha_{i,d})} E_{i,d}, \quad \sigma_d F_{i,d} \sigma_d = (-1)^{p(\alpha_{i,d})} F_{i,d}, \tag{2.3}$$

$$K_{i,d}^{\frac{1}{2}} E_{j,d} K_{i,d}^{-\frac{1}{2}} = q^{(\alpha_{i,d}|\alpha_{j,d})/2} E_{j,d}, \quad K_{i,d}^{\frac{1}{2}} F_{j,d} K_{i,d}^{-\frac{1}{2}} = q^{-(\alpha_{i,d}|\alpha_{j,d})/2} F_{j,d}, \tag{2.4}$$

$$E_{i,d} F_{j,d} - (-1)^{p(\alpha_{i,d})p(\alpha_{j,d})} F_{j,d} E_{i,d} = \delta_{ij} \{ (K_{i,d}^{\frac{1}{2}})^2 - (K_{i,d}^{-\frac{1}{2}})^2 \} / (q - q^{-1}), \tag{2.5}$$

for all $i,\ j \in \mathrm{I}$. The algebra $\mathcal{U}'_d$ has a unique Q_d-grading $\mathcal{U}'_d = \oplus_{\lambda \in Q_d} \mathcal{U}'_{d,\lambda}$, $\mathcal{U}'_{d,\mu}\mathcal{U}'_{d,\lambda} \subset \mathcal{U}'_{d,\mu+\lambda}$ such that $\{1, \sigma_d, K_{i,d}^{\pm\frac{1}{2}}\} \subset \mathcal{U}'_{d,0}$, $E_{i,d} \in \mathcal{U}'_{d,\alpha_{i,d}}$, and $F_{i,d} \in \mathcal{U}'_{d,-\alpha_{i,d}}$ for all $i \in \mathrm{I}$. Then the quantum affine superalgebra

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U'_d of type $D^{(1)}(2,1;x)$ over $\mathbb{C}$ is the quotient algebra of $\mathcal{U}'_d$ divided by the two-sided ideal generated by the following elements:

$$E_{i,d}^2, \quad \text{where } i \in \mathrm{I} \text{ and } p(\alpha_{i,d}) = 1, \tag{2.6}$$
$$[\![E_{i,d}, E_{j,d}]\!], \quad \text{where } i,j \in \mathrm{I},\ i \neq j, \text{ and } (\alpha_{i,d}|\alpha_{j,d}) = 0, \tag{2.7}$$
$$[\![E_{i,d}, [\![E_{i,d}, E_{j,d}]\!]]\!], \quad \text{where } i,j \in \mathrm{I},\ i \neq j, \text{ and } p(\alpha_{i,d}) = 0, \text{ and } (\alpha_{i,d}|\alpha_{j,d}) \neq 0, \tag{2.8}$$
$$[(\alpha_{i,4}|\alpha_{k,4})]_q [\![[\![E_{i,4}, E_{j,4}]\!], E_{k,4}]\!] - [(\alpha_{i,4}|\alpha_{j,4})]_q [\![[\![E_{i,4}, E_{k,4}]\!], E_{j,4}]\!], \quad \text{if } d = 4, \text{ where } i,j,k \in \mathrm{I} \text{ such that } i < j < k, \tag{2.9}$$
$$[(\alpha_{i,d}+\alpha_{d,d}|\alpha_{k,d}+\alpha_{d,d})]_q [\![[\![[\![E_{d,d}, E_{i,d}]\!], [\![E_{d,d}, E_{j,d}]\!]]\!], [\![E_{d,d}, E_{k,d}]\!]]\!] - [(\alpha_{i,d}+\alpha_{d,d}|\alpha_{j,d}+\alpha_{d,d})]_q [\![[\![[\![E_{d,d}, E_{i,d}]\!], [\![E_{d,d}, E_{k,d}]\!]]\!], [\![E_{d,d}, E_{j,d}]\!]]\!] \quad \text{if } d \neq 4, \text{ where } \{i,j,k,d\} = \mathrm{I}, \text{ and } i < j < k, \tag{2.10}$$
$$\Psi_d(X), \quad \text{for all } X \text{ in the above}, \tag{2.11}$$

where the q-super-bracket $[\![\ ,\ ]\!] \colon \mathcal{U}'_d \times \mathcal{U}'_d \to \mathcal{U}'_d$ is a unique bilinear mapping defined by $[\![X_\lambda, X_\mu]\!] := X_\lambda X_\mu - (-1)^{p(\lambda)p(\mu)} q^{-(\lambda|\mu)} X_\mu X_\lambda$ for all $X_\lambda \in \mathcal{U}'_{d,\lambda}$ and $X_\mu \in \mathcal{U}'_{d,\mu}$, and where Ψ_d is a unique algebra automorphism of $\mathcal{U}'_d$ such that $\Psi_d(\sigma_d) = \sigma_d$, $\Psi_d(K_{i,d}^{\pm\frac{1}{2}}) = K_{i,d}^{\mp\frac{1}{2}}$, $\Psi_d(E_{i,d}) = (-1)^{p(\alpha_{i,d})} F_{i,d}$, $\Psi_d(F_{i,d}) = E_{i,d}$.

Let S be a subset of $\mathcal{U}'_d$ consisting of any elements chosen from (2.6)–(2.11), $\langle S \rangle$ the two-sided ideal of $\mathcal{U}'_d$ generated by the elements of S. Then the quotient algebra $A = \mathcal{U}'_d / \langle S \rangle$ has a unique Q_d-grading $A = \oplus_{\lambda \in Q_d} A_\lambda$ induced from the Q_d-grading of $\mathcal{U}'_d$. In particular, the Q_d-grading $U'_d = \oplus_{\lambda \in Q_d} U'_{d,\lambda}$ is induced from that of $\mathcal{U}'_d$. In addition, the tensor algebra $A^{\otimes 2} = A \otimes A$ has a unique Q_d-grading $A^{\otimes 2} = \oplus_{\lambda \in Q_d} A_\lambda^{\otimes 2}$, $A_\lambda^{\otimes 2} A_\mu^{\otimes 2} \subset A_{\lambda+\mu}^{\otimes 2}$ such that $\{1 \otimes 1, \sigma_d \otimes 1, K_{i,d}^{\pm\frac{1}{2}} \otimes 1, 1 \otimes \sigma_d, 1 \otimes K_{i,d}^{\pm\frac{1}{2}}\} \subset A_0^{\otimes 2}$, $\{E_{i,d} \otimes 1, 1 \otimes E_{i,d}\} \subset A_{\alpha_{i,d}}^{\otimes 2}$, and $\{F_{i,d} \otimes 1, 1 \otimes F_{i,d}\} \subset A_{-\alpha_{i,d}}^{\otimes 2}$ for all $i \in \mathrm{I}$. The q-super-brackets $[\![\ ,\ ]\!] \colon A \times A \to A$ and $[\![\ ,\ ]\!] \colon A^{\otimes 2} \times A^{\otimes 2} \to A^{\otimes 2}$ can be defined by the same way as above. We call a non-zero element $x \in A$ (resp. $x \in A^{\otimes 2}$) a *weight vector* with weight λ if $x \in A_\lambda$ (resp. $x \in A_\lambda^{\otimes 2}$), and write $\mathrm{wt}(x) = \lambda$.

For each $\lambda = \frac{1}{2}\sum_{i \in \mathrm{I}} m_i \alpha_{i,d} \in \frac{1}{2} Q_d$ with $m_i \in \mathbb{Z}$, we set $K_\lambda := \Pi_{i \in \mathrm{I}} K_{i,d}^{\frac{1}{2} m_i}$.

Lemma 2.1. *Let A be the quotient algebra of $\mathcal{U}'_d$ divided by the two-sided ideal of $\mathcal{U}'_d$ generated by any elements chosen from (2.6)–(2.11). We assume that x, y, z are weight vectors of A (resp. $A^{\otimes 2}$). Then the following equalities hold in A (resp. $A^{\otimes 2}$):*

$$[\![xy, z]\!] = x[\![y, z]\!] + (-1)^{p(\mathrm{wt}(y))p(\mathrm{wt}(z))} q^{-(\mathrm{wt}(y)|\mathrm{wt}(z))} [\![x, z]\!] y, \tag{2.12}$$
$$[\![x, yz]\!] = [\![x, y]\!] z + (-1)^{p(\mathrm{wt}(x))p(\mathrm{wt}(y))} q^{-(\mathrm{wt}(x)|\mathrm{wt}(y))} y [\![x, z]\!]. \tag{2.13}$$

Moreover, the following equalities hold in $A^{\otimes 2}$:

$$\begin{aligned} &[\![x\sigma_d^{p(\mathrm{wt}(y))} K_{\mathrm{wt}(y)} \otimes y, z\sigma_d^{p(\mathrm{wt}(w))} K_{\mathrm{wt}(w)} \otimes w]\!] \\ &= (-1)^{p(\mathrm{wt}(y))p(\mathrm{wt}(z))} \Big\{ (q - q^{-1})[(\mathrm{wt}(y)|\mathrm{wt}(z))]_q xz + q^{-(\mathrm{wt}(y)|\mathrm{wt}(z))} [\![x, z]\!] \Big\} \sigma_d^{p(\mathrm{wt}(yw))} K_{\mathrm{wt}(yw)} \otimes yw \\ &\qquad + (-1)^{p(\mathrm{wt}(xy))p(\mathrm{wt}(z))} q^{-(\mathrm{wt}(xy)|\mathrm{wt}(z))} zx\sigma_d^{p(\mathrm{wt}(yw))} K_{\mathrm{wt}(yw)} \otimes [\![y, w]\!]. \end{aligned} \tag{2.14}$$

In particular,

$$[\![x \otimes 1, \sigma_d^{\mathrm{wt}(w)} K_{\mathrm{wt}(w)} \otimes w]\!] = 0, \tag{2.15}$$
$$[\![x \otimes 1, z\sigma_d^{p(\mathrm{wt}(w))} K_{\mathrm{wt}(w)} \otimes w]\!] = [\![x, z]\!] \sigma_d^{p(\mathrm{wt}(w))} K_{\mathrm{wt}(w)} \otimes w, \tag{2.16}$$
$$[\![x\sigma_d^{p(\mathrm{wt}(y))} K_{\mathrm{wt}(y)} \otimes y, \sigma_d^{p(\mathrm{wt}(w))} K_{\mathrm{wt}(w)} \otimes w]\!] = x\sigma_d^{p(\mathrm{wt}(yw))} K_{\mathrm{wt}(yw)} \otimes [\![y, w]\!], \tag{2.17}$$
$$[\![\sigma_d^{\mathrm{wt}(y)} K_{\mathrm{wt}(y)} \otimes y, z \otimes 1]\!] = (-1)^{p(\mathrm{wt}(y))p(\mathrm{wt}(z))} (q - q^{-1})[(\mathrm{wt}(y)|\mathrm{wt}(z))]_q z\sigma_d^{\mathrm{wt}(y)} K_{\mathrm{wt}(y)} \otimes y, \tag{2.18}$$
$$\begin{aligned} &[\![x\sigma_d^{p(\mathrm{wt}(y))} K_{\mathrm{wt}(y)} \otimes y, z \otimes 1]\!] \\ &= (-1)^{p(\mathrm{wt}(y))p(\mathrm{wt}(z))} \Big\{ (q - q^{-1})[(\mathrm{wt}(y)|\mathrm{wt}(z))]_q xz + q^{-(\mathrm{wt}(y)|\mathrm{wt}(z))} [\![x, z]\!] \Big\} \sigma_d^{p(\mathrm{wt}(y))} K_{\mathrm{wt}(y)} \otimes y. \end{aligned} \tag{2.19}$$

Proof. We see that

$$\begin{aligned}
&[\![xy, z]\!] = xyz - (-1)^{p(\mathrm{wt}(xy))p(\mathrm{wt}z)}q^{-(\mathrm{wt}(xy)|\mathrm{wt}z)}zxy,\\
&x[\![y, z]\!] = xyz - (-1)^{p(\mathrm{wt}y)p(\mathrm{wt}z)}q^{-(\mathrm{wt}y|\mathrm{wt}z)}xzy,\\
&(-1)^{p(\mathrm{wt}y)p(\mathrm{wt}z)}q^{-(\mathrm{wt}y|\mathrm{wt}z)}[\![x, z]\!]y\\
&\quad = (-1)^{p(\mathrm{wt}y)p(\mathrm{wt}z)}q^{-(\mathrm{wt}y|\mathrm{wt}z)}xzy - (-1)^{p(\mathrm{wt}(xy))p(\mathrm{wt}z)}q^{-(\mathrm{wt}(xy)|\mathrm{wt}z)}zxy,
\end{aligned}$$

which are imply (2.12). We see that

$$\begin{aligned}
&[\![x, yz]\!] = xyz - (-1)^{p(\mathrm{wt}x)p(\mathrm{wt}(yz))}q^{-(\mathrm{wt}x|\mathrm{wt}(yz))}yzx,\\
&[\![x, y]\!]z = xyz - (-1)^{p(\mathrm{wt}x)p(\mathrm{wt}y)}q^{-(\mathrm{wt}x|\mathrm{wt}y)}yxz,\\
&(-1)^{p(\mathrm{wt}x)p(\mathrm{wt}y)}q^{-(\mathrm{wt}x|\mathrm{wt}y)}y[\![x, z]\!]\\
&\quad = (-1)^{p(\mathrm{wt}x)p(\mathrm{wt}y)}q^{-(\mathrm{wt}x|\mathrm{wt}y)}yxz - (-1)^{p(\mathrm{wt}(x))p(\mathrm{wt}(yz))}q^{-(\mathrm{wt}x|\mathrm{wt}(yz))}yzx,
\end{aligned}$$

which are imply (2.13). By the definition of the q-super-bracket, we see that

$$\begin{aligned}
&[\![x\otimes 1, \sigma_d^{\mathrm{wt}(w)}K_{\mathrm{wt}(w)}\otimes w]\!] = x\sigma_d^{\mathrm{wt}(w)}K_{\mathrm{wt}(w)}\otimes w - (-1)^{p(\mathrm{wt}(x))p(\mathrm{wt}(w))}q^{-(\mathrm{wt}(x)|\mathrm{wt}(w))}\sigma_d^{\mathrm{wt}(w)}K_{\mathrm{wt}(w)}x\otimes w = 0,\\
&[\![\sigma_d^{\mathrm{wt}(y)}K_{\mathrm{wt}(y)}\otimes y, z\otimes 1]\!] = \sigma_d^{\mathrm{wt}(y)}K_{\mathrm{wt}(y)}z\otimes y - (-1)^{p(\mathrm{wt}(z))p(\mathrm{wt}(y))}q^{-(\mathrm{wt}(z)|\mathrm{wt}(y))}x\sigma_d^{\mathrm{wt}(y)}K_{\mathrm{wt}(y)}\otimes y\\
&\quad = (-1)^{p(\mathrm{wt}(z))p(\mathrm{wt}(y))}(q^{(\mathrm{wt}(z)|\mathrm{wt}(y))} - q^{-(\mathrm{wt}(z)|\mathrm{wt}(y))})x\sigma_d^{\mathrm{wt}(y)}K_{\mathrm{wt}(y)}\otimes y\\
&\quad = (-1)^{p(\mathrm{wt}(z))p(\mathrm{wt}(y))}(q-q^{-1})[(\mathrm{wt}(z)|\mathrm{wt}(y))]_q z\sigma_d^{\mathrm{wt}(y)}K_{\mathrm{wt}(y)}\otimes y.
\end{aligned}$$

Hence, by (2.13) and (2.12) with the previous equalities, we see that

$$\begin{aligned}
&[\![x\sigma_d^{p(\mathrm{wt}(y))}K_{\mathrm{wt}(y)}\otimes y, z\sigma_d^{p(\mathrm{wt}(w))}K_{\mathrm{wt}(w)}\otimes w]\!]\\
&= [\![x\sigma_d^{p(\mathrm{wt}(y))}K_{\mathrm{wt}(y)}\otimes y, z\otimes 1]\!](\sigma_d^{p(\mathrm{wt}(w))}K_{\mathrm{wt}(w)}\otimes w)\\
&\qquad + (-1)^{p(\mathrm{wt}(xy))p(\mathrm{wt}(z))}q^{-(\mathrm{wt}(xy)|\mathrm{wt}(z))}(z\otimes 1)[\![x\sigma_d^{p(\mathrm{wt}(y))}K_{\mathrm{wt}(y)}\otimes y, \sigma_d^{p(\mathrm{wt}(w))}K_{\mathrm{wt}(w)}\otimes w]\!]\\
&= (-1)^{p(\mathrm{wt}(y))p(\mathrm{wt}(z))}\Big\{(q-q^{-1})[(\mathrm{wt}(y)|\mathrm{wt}(z))]_q xz\sigma_d^{\mathrm{wt}(y)}K_{\mathrm{wt}(y)}\otimes y + q^{-(\mathrm{wt}(y)|\mathrm{wt}(z))}[\![x,z]\!]\sigma_d^{p(\mathrm{wt}(y))}K_{\mathrm{wt}(y)}\otimes y\Big\}\\
&\qquad \times(\sigma_d^{p(\mathrm{wt}(w))}K_{\mathrm{wt}(w)}\otimes w) + (-1)^{p(\mathrm{wt}(xy))p(\mathrm{wt}(z))}q^{-(\mathrm{wt}(xy)|\mathrm{wt}(z))}zx\sigma_d^{p(\mathrm{wt}(yw))}K_{\mathrm{wt}(yw)}\otimes [\![y,w]\!]\\
&= (-1)^{p(\mathrm{wt}(y))p(\mathrm{wt}(z))}\Big\{(q-q^{-1})[(\mathrm{wt}(y)|\mathrm{wt}(z))]_q xz + q^{-(\mathrm{wt}(y)|\mathrm{wt}(z))}[\![x,z]\!]\Big\}\sigma_d^{p(\mathrm{wt}(yw))}K_{\mathrm{wt}(yw)}\otimes yw\\
&\qquad + (-1)^{p(\mathrm{wt}(xy))p(\mathrm{wt}(z))}q^{-(\mathrm{wt}(xy)|\mathrm{wt}(z))}zx\sigma_d^{p(\mathrm{wt}(yw))}K_{\mathrm{wt}(yw)}\otimes [\![y,w]\!]. \qquad \square
\end{aligned}$$

In the following we use notations $E_{\alpha_{i,d}} := E_{i,d}$, $F_{\alpha_{i,d}} := F_{i,d}$.

Lemma 2.2. *Let $\alpha, \beta \in \Pi_d$ with $p(\alpha) = 1$ and $p(\beta) = 0$. Then $\big[\!\!\big[E_\alpha, [\![E_\alpha, E_\beta]\!]\big]\!\!\big] = 0$ in the quotient algebra $\mathcal{U}_d'^{>0}/\langle(2.6)\rangle$ of $\mathcal{U}_d'^{>0}$ divided by the two-sided ideal $\langle(2.6)\rangle$ generated by the elements displayed in (2.6).*

Proof. Since $E_\alpha^2 = 0$ and $(\alpha|\alpha) = 0$, by Lemma 4.3 of [1] with the notation (3.6) of [2], we see that

$$\begin{aligned}
\big[\!\!\big[E_\alpha, [\![E_\alpha, E_\beta]\!]\big]\!\!\big] &= [\![[\![E_\alpha, E_\alpha]\!], E_\beta]\!] + (-1)^{p(\alpha)p(\alpha)}q^{-(\alpha|\alpha)}[E_\alpha, [\![E_\alpha, E_\beta]\!]]_{q^{(\alpha|\alpha-\beta)}}\\
&= 0 - [E_\alpha, [\![E_\alpha, E_\beta]\!]]_{q^{-(\alpha|\alpha+\beta)}} = -[\![E_\alpha, [\![E_\alpha, E_\beta]\!]]\!],
\end{aligned}$$

which implies the required equality. $\square$

Lemma 2.3. *Let $d \neq 4$ and $i \in \mathrm{I}\setminus\{d\}$, and put $E_l = E_{l,d}$ and $\alpha_l = \alpha_{l,d}$ for each $l \in \mathrm{I}$. Then the following equalities hold in $\mathcal{U}_d'^{>0}/\langle(2.6)\rangle$:*

$$[\![E_d, E_i]\!]E_d = -q^{(\alpha_d|\alpha_i)}E_d[\![E_d, E_i]\!], \tag{2.20}$$

$$\big[\!\!\big[[\![E_d, E_i]\!], E_d\big]\!\!\big] = -(q-q^{-1})[(\alpha_d|\alpha_i)]_q E_d[\![E_d, E_i]\!]. \tag{2.21}$$

Moreover, if $\{i,j,k,d\} = \mathrm{I}$, then the following equalities hold in $\mathcal{U}_d'^{>0}/\langle(2.6)\rangle$:

$$\big[\!\!\big[[\![E_d, E_i]\!], [\![E_d, E_j]\!]\big]\!\!\big]E_d = q^{-(\alpha_d|\alpha_k)}E_d\big[\!\!\big[[\![E_d, E_i]\!], [\![E_d, E_j]\!]\big]\!\!\big], \tag{2.22}$$

$$\Big[\!\!\Big[\big[\!\!\big[[\![E_d, E_i]\!], [\![E_d, E_j]\!]\big]\!\!\big], E_d\Big]\!\!\Big] = -(q-q^{-1})[(\alpha_d|\alpha_k)]_q E_d\big[\!\!\big[[\![E_d, E_i]\!], [\![E_d, E_j]\!]\big]\!\!\big]. \tag{2.23}$$

Proof. By Lemma 2.2, we have $\llbracket E_d, \llbracket E_d, E_i \rrbracket \rrbracket = 0$, i.e.,

$$E_d \llbracket E_d, E_i \rrbracket - (-1)^{1\cdot 1} q^{-(\alpha_d|\alpha_d+\alpha_i)} \llbracket E_d, E_i \rrbracket E_d = 0,$$

which implies (2.20). By (2.20), we see that

$$\llbracket \llbracket E_d, E_i \rrbracket, E_d \rrbracket = \llbracket E_d, E_i \rrbracket E_d - (-1)^{1\cdot 1} q^{-(\alpha_d|\alpha_i)} E_d \llbracket E_d, E_i \rrbracket = -(q-q^{-1})[(\alpha_d|\alpha_i)]_q E_d \llbracket E_d, E_i \rrbracket.$$

Moreover, it is easy to see that

$$0 = \Big\llbracket E_d, \llbracket \llbracket E_d, E_i \rrbracket, \llbracket E_d, E_j \rrbracket \rrbracket \Big\rrbracket = E_d \llbracket \llbracket E_d, E_i \rrbracket, \llbracket E_d, E_j \rrbracket \rrbracket - (-1)^{1\cdot 2} q^{-(\alpha_d|\alpha_i+\alpha_j)} \llbracket \llbracket E_d, E_i \rrbracket, \llbracket E_d, E_j \rrbracket \rrbracket E_d,$$

which implies (2.22) since $(\alpha_d|\alpha_i+\alpha_j+\alpha_k) = 0$. By (2.22), we see that

$$\begin{aligned} \Big\llbracket \llbracket \llbracket E_d, E_i \rrbracket, \llbracket E_d, E_j \rrbracket \rrbracket, E_d \Big\rrbracket &= \llbracket \llbracket E_d, E_i \rrbracket, \llbracket E_d, E_j \rrbracket \rrbracket E_d - (-1)^{2\cdot 1} q^{-(\alpha_d|\alpha_i+\alpha_j)} E_d \llbracket \llbracket E_d, E_i \rrbracket, \llbracket E_d, E_j \rrbracket \rrbracket \\ &= \llbracket \llbracket E_d, E_i \rrbracket, \llbracket E_d, E_j \rrbracket \rrbracket E_d - q^{(\alpha_d|\alpha_k)} E_d \llbracket \llbracket E_d, E_i \rrbracket, \llbracket E_d, E_j \rrbracket \rrbracket \\ &= -(q-q^{-1})[(\alpha_d|\alpha_k)]_q E_d \llbracket \llbracket E_d, E_i \rrbracket, \llbracket E_d, E_j \rrbracket \rrbracket. \quad \square \end{aligned}$$

Lemma 2.4. *Let $d \neq 4$ and $i, j \in \mathrm{I} \setminus \{d\}$ with $i \neq j$, and put $E_l = E_{l,d}$ and $\alpha_l = \alpha_{l,d}$ for each $l \in \mathrm{I}$. Then*

$$\llbracket \llbracket E_d, E_i \rrbracket, E_j \rrbracket = \llbracket \llbracket E_d, E_j \rrbracket, E_i \rrbracket \tag{2.24}$$

in the quotient algebra $\mathcal{U}_d'^{>0}/\langle(2.7)\rangle$ of $\mathcal{U}_d'^{>0}$ divided by the two-sided ideal $\langle(2.7)\rangle$ generated by the elements displayed in (2.7). Moreover, if $\{i, j, k, d\} = \mathrm{I}$, then the following equality hold in $\mathcal{U}_d'^{>0}/\langle(2.7)\rangle$:

$$\begin{aligned} \Big\llbracket \llbracket \llbracket E_d, E_i \rrbracket, \llbracket E_d, E_j \rrbracket \rrbracket, E_k \Big\rrbracket &= \Big\llbracket \llbracket \llbracket E_d, E_i \rrbracket, \llbracket \llbracket E_d, E_j \rrbracket, E_k \rrbracket \Big\rrbracket + q^{-(\alpha_d|\alpha_k)} \llbracket E_d, E_k \rrbracket E_i \llbracket E_d, E_j \rrbracket \\ &- q^{(\alpha_d|\alpha_j)} E_i \llbracket E_d, E_k \rrbracket \llbracket E_d, E_j \rrbracket + q^{(\alpha_d|\alpha_k)} \llbracket E_d, E_j \rrbracket \llbracket E_d, E_k \rrbracket E_i - q^{(\alpha_d|\alpha_k-\alpha_1)} \llbracket E_d, E_j \rrbracket E_i \llbracket E_d, E_k \rrbracket. \end{aligned} \tag{2.25}$$

Proof. Since $\llbracket E_i, E_j \rrbracket = 0$, by Lemma 4.3 of [1] with the notation (3.6) of [2], we see that

$$\llbracket \llbracket E_d, E_i \rrbracket, E_j \rrbracket = 0 + \big[\llbracket E_d, E_j \rrbracket, E_i \big]_{q^{(\alpha_i|\alpha_j-\alpha_d)}} = \big[\llbracket E_d, E_j \rrbracket, E_i \big]_{q^{-(\alpha_d|\alpha_i)}} = \llbracket \llbracket E_d, E_j \rrbracket, E_j \rrbracket$$

and that

$$\begin{aligned} &\Big\llbracket \llbracket \llbracket E_d, E_i \rrbracket, \llbracket E_d, E_j \rrbracket \rrbracket, E_k \Big\rrbracket = \Big\llbracket \llbracket \llbracket E_d, E_i \rrbracket, \llbracket \llbracket E_d, E_j \rrbracket, E_k \rrbracket \Big\rrbracket + q^{-(\alpha_d|\alpha_k)} \Big[\llbracket \llbracket E_d, E_i \rrbracket, E_k \rrbracket, \llbracket E_d, E_j \rrbracket \Big]_{q^{(\alpha_d+\alpha_j|\alpha_k-\alpha_d-\alpha_i)}} \\ &= \Big\llbracket \llbracket \llbracket E_d, E_i \rrbracket, \llbracket \llbracket E_d, E_j \rrbracket, E_k \rrbracket \Big\rrbracket + q^{-(\alpha_d|\alpha_k)} \Big[\llbracket \llbracket E_d, E_i \rrbracket, E_k \rrbracket, \llbracket E_d, E_j \rrbracket \Big]_{q^{(\alpha_d|\alpha_k-\alpha_i-\alpha_j)}} \\ &= \Big\llbracket \llbracket \llbracket E_d, E_i \rrbracket, \llbracket \llbracket E_d, E_j \rrbracket, E_k \rrbracket \Big\rrbracket + q^{-(\alpha_d|\alpha_k)} \Big[\llbracket \llbracket E_d, E_k \rrbracket, E_i \rrbracket, \llbracket E_d, E_j \rrbracket \Big]_{q^{2(\alpha_d|\alpha_k)}} \\ &= \Big\llbracket \llbracket \llbracket E_d, E_i \rrbracket, \llbracket \llbracket E_d, E_j \rrbracket, E_k \rrbracket \Big\rrbracket + q^{-(\alpha_d|\alpha_k)} \llbracket \llbracket E_d, E_k \rrbracket, E_i \rrbracket \llbracket E_d, E_j \rrbracket + q^{(\alpha_d|\alpha_k)} \llbracket E_d, E_j \rrbracket \llbracket \llbracket E_d, E_k \rrbracket, E_i \rrbracket \\ &= \Big\llbracket \llbracket \llbracket E_d, E_i \rrbracket, \llbracket \llbracket E_d, E_j \rrbracket, E_k \rrbracket \Big\rrbracket + q^{-(\alpha_d|\alpha_k)} (\llbracket E_d, E_k \rrbracket E_i - q^{-(\alpha_d|\alpha_i)} E_i \llbracket E_d, E_k \rrbracket) \llbracket E_d, E_j \rrbracket \\ &\qquad + q^{(\alpha_d|\alpha_k)} \llbracket E_d, E_j \rrbracket (\llbracket E_d, E_k \rrbracket E_i - q^{-(\alpha_d|\alpha_i)} E_i \llbracket E_d, E_k \rrbracket) \\ &= \Big\llbracket \llbracket \llbracket E_d, E_i \rrbracket, \llbracket \llbracket E_d, E_j \rrbracket, E_k \rrbracket \Big\rrbracket + q^{-(\alpha_d|\alpha_k)} \llbracket E_d, E_k \rrbracket E_i \llbracket E_d, E_j \rrbracket - q^{(\alpha_d|\alpha_j)} E_i \llbracket E_d, E_k \rrbracket \llbracket E_d, E_j \rrbracket \\ &\qquad + q^{(\alpha_d|\alpha_k)} \llbracket E_d, E_j \rrbracket \llbracket E_d, E_k \rrbracket E_i - q^{(\alpha_d|\alpha_k-\alpha_i)} \llbracket E_d, E_j \rrbracket E_i \llbracket E_d, E_k \rrbracket. \quad \square \end{aligned}$$

3 Hopf algebra structures

In this section, we will construct Hopf algebra structures on the quantum affine superalgebras of type $D^{(1)}(2,1;x)$.

Let us recall the definition of the Hopf algebra. Let A be an associative algebra over a field $\mathbb{K}$ with the unit 1_A, $\Delta\colon A \to A \otimes_{\mathbb{K}} A$ an algebra homomorphism, $\varepsilon\colon A \to \mathbb{K}$ an algebra homomorphism, and $S\colon A \to A$ an algebra anti-homomorphism such that

$$(\Delta \otimes id_A) \circ \Delta = (id_A \otimes \Delta) \circ \Delta, \tag{3.1}$$

$$(\varepsilon \otimes id_A) \circ \Delta = id_A = (id_A \otimes \varepsilon) \circ \Delta, \tag{3.2}$$

$$m \circ (S \otimes id_A) \circ \Delta = \iota \circ \varepsilon = m \circ (id_A \otimes S) \circ \Delta, \tag{3.3}$$

where $m\colon A \otimes_{\mathbb{K}} A \to A$ is the multiplication $m(a \otimes a') = aa'$ for all $a, a' \in A$, and where $\iota\colon \mathbb{K} \to A$ is the embedding $\iota(k) = k1_A$ for all $k \in \mathbb{K}$. Then the quadruplet $(A, \Delta, \varepsilon, S)$ is called a *Hopf algebra* over $\mathbb{K}$.

Proposition 3.1 ([1]). *The associative $\mathbb{C}$-algebra $\mathcal{U}'_d$ equipped with the following (Δ, ε, S) is a Hopf algebra:*

$$\Delta(X) = X \otimes X, \quad \Delta(E_{i,d}) = E_{i,d} \otimes 1 + K_{i,d}\sigma_d^{p(\alpha_{i,d})} \otimes E_{i,d}, \quad \Delta(F_{i,d}) = F_{i,d} \otimes K_{i,d}^{-1} + \sigma_d^{p(\alpha_{i,d})} \otimes F_{i,d}, \tag{3.4}$$

$$\varepsilon(X) = 1, \quad \varepsilon(E_{i,d}) = 0, \quad \varepsilon(F_{i,d}) = 0, \tag{3.5}$$

$$S(X) = X^{-1}, \quad S(E_{i,d}) = -K_{i,d}^{-1}\sigma_d^{p(\alpha_{i,d})}E_{i,d}, \quad S(F_{i,d}) = -(-1)^{p(\alpha_{i,d})}F_{i,d}K_{i,d}\sigma_d^{p(\alpha_{i,d})}, \tag{3.6}$$

where $i \in \mathrm{I}$ and $X \in \{\sigma_d,\ K_{i,d}^{\pm\frac{1}{2}} \mid i \in \mathrm{I}\}$.

Proof. To prove the existence of the homomorphisms Δ and ε and the anti-homomorphism S, it suffices to check that the images of the generators under Δ, ε, and S satisfy (2.1)–(2.5). Here we check only (2.5). Set $\alpha = \alpha_{i,d}$ and $\beta = \alpha_{j,d}$. Then we see that

$$\begin{aligned}
&\Delta(E_\alpha)\Delta(F_\beta) - (-1)^{p(\alpha)p(\beta)}\Delta(F_\beta)\Delta(E_\alpha) = (E_\alpha \otimes 1 + K_\alpha\sigma_d^{p(\alpha)} \otimes E_\alpha)(F_\beta \otimes K_\beta^{-1} + \sigma_d^{p(\beta)} \otimes F_\beta) \\
&\quad - (-1)^{p(\alpha)p(\beta)}(F_\beta \otimes K_\beta^{-1} + \sigma_d^{p(\beta)} \otimes F_\beta)(E_\alpha \otimes 1 + K_\alpha\sigma_d^{p(\alpha)} \otimes E_\alpha) \\
&\quad = \{E_\alpha F_\beta - (-1)^{p(\alpha)p(\beta)}F_\beta E_\alpha\} \otimes K_\beta^{-1} + K_\alpha\sigma_d^{p(\alpha+\beta)} \otimes \{E_\alpha F_\beta - (-1)^{p(\alpha)p(\beta)}F_\beta E_\alpha\} \\
&\quad = \delta_{\beta,\alpha}\{(K_\alpha - K_\alpha^{-1})/(q-q^{-1}) \otimes K_\alpha^{-1} + K_\alpha \otimes (K_\alpha - K_\alpha^{-1})/(q-q^{-1})\} \\
&\quad = \delta_{\beta,\alpha}(K_\alpha \otimes K_\alpha - K_\alpha^{-1} \otimes K_\alpha^{-1})/(q-q^{-1}) = \delta_{\beta,\alpha}\{\Delta(K_\alpha) - \Delta(K_\alpha^{-1})\}/(q-q^{-1}), \\
&\varepsilon(E_\alpha)\varepsilon(F_\beta) - (-1)^{p(\alpha)p(\beta)}\varepsilon(F_\beta)\varepsilon(E_\alpha) = 0 = \delta_{\beta,\alpha}\{\varepsilon(K_\alpha) - \varepsilon(K_\alpha^{-1})\}/(q-q^{-1}), \\
&S(F_\beta)S(E_\alpha) - (-1)^{p(\alpha)p(\beta)}S(E_\alpha)S(F_\beta) \\
&\quad = (-1)^{p(\beta)}F_\beta K_\beta\sigma_d^{p(\beta)}K_\alpha^{-1}\sigma_d^{p(\alpha)}E_\alpha - (-1)^{p(\alpha)p(\beta)}K_\alpha^{-1}\sigma_d^{p(\alpha)}E_\alpha(-1)^{p(\beta)}F_\beta K_\beta\sigma_d^{p(\beta)} \\
&\quad = (-1)^{p(\alpha+\beta)+p(\alpha)p(\beta)}q^{(\alpha|\beta-\alpha)}\{F_\beta E_\alpha - (-1)^{p(\alpha)p(\beta)}E_\alpha F_\beta\}K_{\beta-\alpha}\sigma_d^{p(\alpha+\beta)} \\
&\quad = -\delta_{\beta,\alpha}(K_\alpha - K_\alpha^{-1})/(q-q^{-1}) = \delta_{\beta,\alpha}\{S(K_\alpha) - S(K_\alpha^{-1})\}/(q-q^{-1}).
\end{aligned}$$

Thus the images of the generators under Δ, ε, and S satisfy (2.5).

We next check the equalities (3.1)–(3.3). Since Δ and ε are homomorphisms and S is an anti-homomorphism, it suffices to show that the equalities (3.1)–(3.3) are valid for the generators. Here we check only for E_α with $\alpha \in \Pi_d$. We see that

$$\begin{aligned}
&(\Delta \otimes id_A) \circ \Delta(E_\alpha) = (\Delta \otimes id_A)(E_\alpha \otimes 1 + K_\alpha\sigma_d^{p(\alpha)} \otimes E_\alpha) \\
&\quad = (E_\alpha \otimes 1 + K_\alpha\sigma_d^{p(\alpha)} \otimes E_\alpha) \otimes 1 + K_\alpha\sigma_d^{p(\alpha)} \otimes K_\alpha\sigma_d^{p(\alpha)} \otimes E_\alpha, \\
&(id_A \otimes \Delta) \circ \Delta(E_\alpha) = (id_A \otimes \Delta)(E_\alpha \otimes 1 + K_\alpha\sigma_d^{p(\alpha)} \otimes E_\alpha) \\
&\quad = E_\alpha \otimes 1 \otimes 1 + K_\alpha\sigma_d^{p(\alpha)} \otimes (E_\alpha \otimes 1 + K_\alpha\sigma_d^{p(\alpha)} \otimes E_\alpha), \\
&(\varepsilon \otimes id_A) \circ \Delta(E_\alpha) = (\varepsilon \otimes id_A)(E_\alpha \otimes 1 + K_\alpha\sigma_d^{p(\alpha)} \otimes E_\alpha) = E_\alpha, \\
&(id_A \otimes \varepsilon) \circ \Delta(E_\alpha) = (id_A \otimes \varepsilon)(E_\alpha \otimes 1 + K_\alpha\sigma_d^{p(\alpha)} \otimes E_\alpha) = E_\alpha, \\
&m \circ (S \otimes id_A) \circ \Delta(E_\alpha) = m \circ (S \otimes id_A)(E_\alpha \otimes 1 + K_\alpha\sigma_d^{p(\alpha)} \otimes E_\alpha) \\
&\quad = -K_\alpha^{-1}\sigma_d^{p(\alpha)}E_\alpha + K_\alpha^{-1}\sigma_d^{-p(\alpha)}E_\alpha = 0 = \iota \circ \varepsilon(E_\alpha), \\
&m \circ (id_A \otimes S) \circ \Delta(E_\alpha) = m \circ (id_A \otimes S)(E_\alpha \otimes 1 + K_\alpha\sigma_d^{p(\alpha)} \otimes E_\alpha) \\
&\quad = E_\alpha + K_\alpha\sigma_d^{p(\alpha)} \cdot (-K_\alpha^{-1}\sigma_d^{p(\alpha)}E_\alpha) = 0 = \iota \circ \varepsilon(E_\alpha).
\end{aligned}$$

Thus the equalities (3.1)–(3.3) are valid for the generator E_α with $\alpha \in \Pi_d$. □

Lemma 3.2. *For each $X_\mu \in \mathcal{U}'_{d,\mu}$ and $X_\nu \in \mathcal{U}'_{d,\nu}$, the following equality holds:*

$$\Delta([\![X_\mu, X_\nu]\!]) = [\![\Delta(X_\mu), \Delta(X_\nu)]\!]. \tag{3.7}$$

Proof. Since $\Delta(X_\mu) \in \mathcal{U}'^{\otimes 2}_{d,\mu}$ and $\Delta(X_\nu) \in \mathcal{U}'^{\otimes 2}_{d,\nu}$, we have

$$\Delta([\![X_\mu, X_\nu]\!]) = \Delta(X_\mu)\Delta(X_\nu) - (-1)^{p(\mu)p(\nu)}q^{-(\mu|\nu)}\Delta(X_\nu)\Delta(X_\mu) = [\![\Delta(X_\mu), \Delta(X_\nu)]\!]. \quad \square$$

From now on, we will give several formulas concerning the coproduct Δ of $\mathcal{U}'_d$.

Let $\alpha \in \Pi_d$ such that $p(\alpha)=1$. Then $(\alpha|\alpha)=0$. We see that

$$\begin{aligned}\Delta(E_\alpha^2) &= \Delta(E_\alpha)^2 = (E_\alpha \otimes 1 + \sigma_d K_\alpha \otimes E_\alpha)^2 \\ &= E_\alpha^2 \otimes 1 + E_\alpha \sigma_d K_\alpha \otimes E_\alpha + \sigma_d K_\alpha E_\alpha \otimes E_\alpha + \sigma_d^2 K_\alpha^2 \otimes E_\alpha^2 \\ &= E_\alpha^2 \otimes 1 + \{1-q^{(\alpha|\alpha)}\} E_\alpha \sigma_d K_\alpha \otimes E_\alpha + K_\alpha^2 \otimes E_\alpha^2 = E_\alpha^2 \otimes 1 + K_\alpha^2 \otimes E_\alpha^2. \end{aligned} \tag{3.8}$$

Let $\alpha, \beta \in \Pi_d$ with $\alpha \neq \beta$. By Lemma 3.2 and Lemma 2.1, we see that

$$\Delta([\![E_\alpha, E_\beta]\!]) = [\![E_\alpha, E_\beta]\!] \otimes 1 + (-1)^{p(\alpha)p(\beta)}(q-q^{-1})[(\alpha|\beta)]_q E_\beta \sigma_d^{p(\alpha)} K_\alpha \otimes E_\alpha + \sigma_d^{p(\alpha+\beta)} K_{\alpha+\beta} \otimes [\![E_\alpha, E_\beta]\!]. \tag{3.9}$$

In particular, if $(\alpha|\beta)=0$, then we have

$$\Delta([\![E_\alpha, E_\beta]\!]) = [\![E_\alpha, E_\beta]\!] \otimes 1 + \sigma_d^{p(\alpha+\beta)} K_{\alpha+\beta} \otimes [\![E_\alpha, E_\beta]\!]. \tag{3.10}$$

In particular, if $p(\alpha)=0$, then we have

$$\Delta([\![E_\alpha, E_\beta]\!]) = [\![E_\alpha, E_\beta]\!] \otimes 1 + (q^{(\alpha|\beta)} - q^{-(\alpha|\beta)}) E_\beta K_\alpha \otimes E_\alpha + \sigma_d^{p(\beta)} K_{\alpha+\beta} \otimes [\![E_\alpha, E_\beta]\!].$$

Let $\alpha, \beta \in \Pi_d$ with $\alpha \neq \beta$ such that $p(\alpha)=0$ and $(\alpha|\beta) \neq 0$. Then we note that $(\alpha|\alpha+2\beta)=0$. We have

$$[\![E_\alpha, E_\beta]\!] = E_{(\alpha,\beta)} - q^{-(\alpha|\beta)} E_{(\beta,\alpha)}, \qquad [\![E_\alpha, [\![E_\alpha, E_\beta]\!]]\!] = E_\alpha [\![E_\alpha, E_\beta]\!] - q^{-(\alpha|\alpha+\beta)} [\![E_\alpha, E_\beta]\!] E_\alpha.$$

By Lemma 3.2 and (3.9), we see that

$$\begin{aligned}&\Delta([\![E_\alpha, [\![E_\alpha, E_\beta]\!]]\!]) = [\![\Delta(E_\alpha), \Delta([\![E_\alpha, E_\beta]\!])]\!] \\ &= [\![(E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha),\ [\![E_\alpha, E_\beta]\!] \otimes 1 + (q-q^{-1})[(\alpha|\beta)]_q E_\beta K_\alpha \otimes E_\alpha + \sigma_d^{p(\beta)} K_{\alpha+\beta} \otimes [\![E_\alpha, E_\beta]\!]]\!] \\ &= [\![E_\alpha, [\![E_\alpha, E_\beta]\!]]\!] \otimes 1 + (q-q^{-1})[(\alpha|\beta)]_q [\![E_\alpha \otimes 1, E_\beta K_\alpha \otimes E_\alpha]\!] + [\![E_\alpha \otimes 1, \sigma_d^{p(\beta)} K_{\alpha+\beta} \otimes [\![E_\alpha, E_\beta]\!]]\!] \\ &+ [\![K_\alpha \otimes E_\alpha, [\![E_\alpha, E_\beta]\!] \otimes 1]\!] + (q-q^{-1})[(\alpha|\beta)]_q [\![K_\alpha \otimes E_\alpha, E_\beta K_\alpha \otimes E_\alpha]\!] + \sigma_d^{p(\beta)} K_{2\alpha+\beta} \otimes [\![E_\alpha, [\![E_\alpha, E_\beta]\!]]\!].\end{aligned}$$

Here, by Lemma 2.1 we see that

$$\begin{aligned}&[\![E_\alpha \otimes 1, \sigma_d^{p(\beta)} K_{\alpha+\beta} \otimes [\![E_\alpha, E_\beta]\!]]\!] = 0, \\ &[\![K_\alpha \otimes E_\alpha, E_\beta K_\alpha \otimes E_\alpha]\!] = K_\alpha E_\beta K_\alpha \otimes E_\alpha^2 - q^{-(\alpha|\alpha+\beta)} E_\beta K_\alpha^2 \otimes E_\alpha^2 \\ &\qquad = q^{(\alpha|\beta)}(1-q^{-(\alpha|\alpha+2\beta)}) E_\beta K_\alpha^2 \otimes E_\alpha^2 = 0 \qquad (\because (\alpha|\alpha+2\beta)=0), \\ &(q-q^{-1})[(\alpha|\beta)]_q [\![E_\alpha \otimes 1, E_\beta K_\alpha \otimes E_\alpha]\!] + [\![K_\alpha \otimes E_\alpha, [\![E_\alpha, E_\beta]\!] \otimes 1]\!] \\ &\quad = (q-q^{-1})[(\alpha|\beta)]_q [\![E_\alpha, E_\beta]\!] K_\alpha \otimes E_\alpha + (q-q^{-1})[(\alpha|\alpha+\beta)]_q [\![E_\alpha, E_\beta]\!] K_\alpha \otimes E_\alpha \\ &\quad = (q^{(\alpha|\beta)} - q^{-(\alpha|\beta)}) [\![E_\alpha, E_\beta]\!] K_\alpha \otimes E_\alpha + (q^{(\alpha|\alpha+\beta)} - q^{-(\alpha|\alpha+\beta)}) [\![E_\alpha, E_\beta]\!] K_\alpha \otimes E_\alpha \\ &\quad = (q^{(\alpha|\beta)} + q^{(\alpha|\alpha+\beta)})(1-q^{-(\alpha|\alpha+2\beta)}) [\![E_\alpha, E_\beta]\!] K_\alpha \otimes E_\alpha = 0.\end{aligned}$$

Thus, if distinct elements $\alpha, \beta \in \Pi_d$ satisfy $p(\alpha)=0$ and $(\alpha|\beta) \neq 0$, then

$$\Delta([\![E_\alpha, [\![E_\alpha, E_\beta]\!]]\!]) = [\![E_\alpha, [\![E_\alpha, E_\beta]\!]]\!] \otimes 1 + \sigma_d^{p(\beta)} K_{2\alpha+\beta} \otimes [\![E_\alpha, [\![E_\alpha, E_\beta]\!]]\!]. \tag{3.11}$$

We assume that $d=4$ and put $E_l = E_{l,4}$, $K_l = K_{l,4}$, and $\alpha_l = \alpha_{l,4}$ for each $l \in \mathrm{I}$. Then we will check that

$$\begin{aligned}&\Delta(-[(\alpha_i|\alpha_k)]_q [\![[\![E_i, E_j]\!], E_k]\!] + [(\alpha_i|\alpha_j)]_q [\![[\![E_i, E_k]\!], E_j]\!]) \\ &\quad = (-[(\alpha_i|\alpha_k)]_q [\![[\![E_i, E_j]\!], E_k]\!] + [(\alpha_i|\alpha_j)]_q [\![[\![E_i, E_k]\!], E_j]\!]) \otimes 1 \\ &\qquad + \sigma_4 K_i K_j K_k \otimes (-[(\alpha_i|\alpha_k)]_q [\![[\![E_i, E_j]\!], E_k]\!] + [(\alpha_i|\alpha_j)]_q [\![[\![E_i, E_k]\!], E_j]\!])\end{aligned} \tag{3.12}$$

for each $i, j, k \in \mathrm{I}$ with $i<j<k$. Note that $p(\alpha_l)=1$ for all $l \in \mathrm{I}$. By Lemma 3.2 and (3.9), we see that

$$\begin{aligned}&\Delta([\![[\![E_i, E_j]\!], E_k]\!]) = [\![\Delta([\![E_i, E_j]\!]), \Delta(E_k)]\!] \\ &= [\![[\![E_i, E_j]\!] \otimes 1 + (-1)^{p(\alpha_i)p(\alpha_j)}(q-q^{-1})[(\alpha_i|\alpha_j)]_q E_j \sigma_4^{p(\alpha_i)} K_i \otimes E_i + \sigma_4^{p(\alpha_i+\alpha_j)} K_i K_j \otimes [\![E_i, E_j]\!], \\ &\quad E_k \otimes 1 + \sigma_4 K_k \otimes E_k]\!] \\ &= [\![[\![E_i, E_j]\!] \otimes 1, E_k \otimes 1]\!] + [\![[\![E_i, E_j]\!] \otimes 1, \sigma_4 K_k \otimes E_k]\!] \\ &\quad - (q-q^{-1})[(\alpha_i|\alpha_j)]_q [\![E_j \sigma_4 K_i \otimes E_i, E_k \otimes 1]\!] - (q-q^{-1})[(\alpha_i|\alpha_j)]_q [\![E_j \sigma_4 K_i \otimes E_i, \sigma_4 K_k \otimes E_k]\!] \\ &\quad + [\![K_i K_j \otimes [\![E_i, E_j]\!], E_k \otimes 1]\!] + [\![K_i K_j \otimes [\![E_i, E_j]\!], \sigma_4 K_k \otimes E_k]\!].\end{aligned}$$

Here, by Lemma 2.1 we see that

$$\begin{aligned}
&[\![[\![E_i,E_j]\!]\otimes 1, E_k\otimes 1]\!] = [\![[\![E_i,E_j]\!],E_k]\!]\otimes 1, \qquad [\![[\![E_i,E_j]\!]\otimes 1,\sigma_4K_k\otimes E_k]\!]=0,\\
&[\![E_j\sigma_4K_i\otimes E_i, E_k\otimes 1]\!] = (-1)^{1\cdot 1}\{(q-q^{-1})[(\alpha_i|\alpha_k)]_qE_jE_k + q^{-(\alpha_i|\alpha_k)}[\![E_j,E_k]\!]\}\sigma_dK_i\otimes E_i\\
&\qquad\qquad\qquad\qquad\qquad = -\{q^{(\alpha_i|\alpha_k)}E_jE_k + q^{(\alpha_i|\alpha_j)}E_kE_j\}\sigma_dK_i\otimes E_i,\\
&[\![E_j\sigma_4K_i\otimes E_i,\sigma_4K_k\otimes E_k]\!] = E_jK_iK_k\otimes[\![E_i,E_k]\!],\\
&[\![K_iK_j\otimes[\![E_i,E_j]\!],E_k\otimes 1]\!] = (-1)^{2\cdot 1}(q-q^{-1})[(\alpha_i+\alpha_j|\alpha_k)]_qE_kK_iK_j\otimes[\![E_i,E_j]\!],\\
&K_iK_j\otimes[\![[\![E_i,E_j]\!],\sigma_4K_k\otimes E_k]\!] = \sigma_4K_iK_jK_k\otimes[\![[\![E_i,E_j]\!],E_k]\!].
\end{aligned}$$

Thus, by using the equality $(\alpha_i|\alpha_j)+(\alpha_j|\alpha_k)+(\alpha_k|\alpha_i)=0$, we see that

$$\begin{aligned}
&[(\alpha_i|\alpha_k)]_q\Delta([\![[\![E_i,E_j]\!],E_k]\!]) = [(\alpha_i|\alpha_k)]_q[\![[\![E_i,E_j]\!],E_k]\!]\otimes 1\\
&+(q-q^{-1})[(\alpha_i|\alpha_k)]_q[(\alpha_i|\alpha_j)]_q\{q^{(\alpha_i|\alpha_k)}E_jE_k+q^{(\alpha_i|\alpha_j)}E_kE_j\}\sigma_dK_i\otimes E_i,\\
&-(q-q^{-1})[(\alpha_i|\alpha_k)]_q[(\alpha_i|\alpha_j)]_qE_jK_iK_k\otimes[\![E_i,E_k]\!]-(q-q^{-1})[(\alpha_i|\alpha_k)]_q[(\alpha_i|\alpha_j)]_qE_kK_iK_j\otimes[\![E_i,E_j]\!]\\
&+(q-q^{-1})[(\alpha_i|\alpha_k)]_q\sigma_4K_iK_jK_k\otimes[\![[\![E_i,E_j]\!],E_k]\!],
\end{aligned}$$

and hence that

$$\begin{aligned}
&[(\alpha_i|\alpha_j)]_q\Delta([\![[\![E_i,E_k]\!],E_j]\!]) = [(\alpha_i|\alpha_j)]_q[\![[\![E_i,E_k]\!],E_j]\!]\otimes 1\\
&+(q-q^{-1})[(\alpha_i|\alpha_j)]_q[(\alpha_i|\alpha_k)]_q\{q^{(\alpha_i|\alpha_j)}E_kE_j+q^{(\alpha_i|\alpha_k)}E_jE_k\}\sigma_dK_i\otimes E_i,\\
&-(q-q^{-1})[(\alpha_i|\alpha_j)]_q[(\alpha_i|\alpha_k)]_qE_kK_iK_j\otimes[\![E_i,E_j]\!]-(q-q^{-1})[(\alpha_i|\alpha_j)]_q[(\alpha_i|\alpha_k)]_qE_jK_iK_k\otimes[\![E_i,E_k]\!]\\
&+(q-q^{-1})[(\alpha_i|\alpha_j)]_q\sigma_4K_iK_kK_j\otimes[\![[\![E_i,E_k]\!],E_j]\!].
\end{aligned}$$

Therefore (3.12) is valid.

We next assume that $d\neq 4$ and $\{i,j,k,d\}=\mathrm{I}$ with $i<j<k$. We put $E_l=E_{l,d}$, $K_l=K_{l,d}$, and $\alpha_l=\alpha_{l,d}$ for each $l\in\mathrm{I}$. Note that $p(\alpha_d)=1$, $p(\alpha_i)=p(\alpha_j)=p(\alpha_k)=0$ and $(\alpha_d|\alpha_d)=(\alpha_d|\alpha_i+\alpha_j+\alpha_k)=0$. Let $\mathcal{SR}(2.10)\in\mathcal{U}'^{>0}_d$ be an arbitrary element displayed in (2.10). Then we see that

$$\mathcal{SR}(2.10) = -[(\alpha_d|\alpha_j)]_q\Big[\!\!\Big[[\![[\![E_d,E_i]\!],[\![E_d,E_j]\!]]\!],[\![E_d,E_k]\!]\Big]\!\!\Big]+[(\alpha_d|\alpha_k)]_q\Big[\!\!\Big[[\![[\![E_d,E_i]\!],[\![E_d,E_k]\!]]\!],[\![E_d,E_j]\!]\Big]\!\!\Big]. \quad (3.13)$$

Let $\langle(2.6),(2.7)\rangle$ be the two-sided ideal of $\mathcal{U}'^{\geq 0}_d$ generated by the elements displayed in (2.6) and (2.7). Then we will check the following equality in $(\mathcal{U}'^{\geq 0}_d/\langle(2.6),(2.7)\rangle)^{\otimes 2}$:

$$\Delta(\mathcal{SR}(2.10)) = \mathcal{SR}(2.10)\otimes 1+\sigma_dK_d^3K_iK_jK_k\otimes\mathcal{SR}(2.10). \quad (3.14)$$

Here we use the following identification:

$$(\mathcal{U}'^{\geq 0}_d/\langle(2.6),(2.7)\rangle)^{\otimes 2}\simeq(\mathcal{U}'^{\geq 0}_d)^{\otimes 2}/\{\langle(2.6),(2.7)\rangle\otimes\mathcal{U}'^{\geq 0}_d+\mathcal{U}'^{\geq 0}_d\otimes\langle(2.6),(2.7)\rangle\}.$$

For each $l\in\mathrm{I}$ with $l\neq d$, we see that the following equality holds in $(\mathcal{U}'^{\geq 0}_d)^{\otimes 2}$:

$$\Delta([\![E_d,E_l]\!]) = [\![E_d,E_l]\!]\otimes 1+(q-q^{-1})[(\alpha_d|\alpha_l)]_qE_l\sigma_dK_d\otimes E_d+\sigma_dK_dK_l\otimes[\![E_d,E_l]\!]. \quad (3.15)$$

By Lemma 2.1, we have the following equality in $(\mathcal{U}'^{\geq 0}_d/\langle(2.6),(2.7)\rangle)^{\otimes 2}$:

$$\begin{aligned}
&[\![\Delta([\![E_d,E_i]\!]),\Delta([\![E_d,E_j]\!])]\!] = [\![[\![E_d,E_i]\!],[\![E_d,E_j]\!]]\!]\otimes 1+(q-q^{-1})[(\alpha_d|\alpha_k)]_q[\![E_i,[\![E_d,E_j]\!]]\!]\sigma_dK_d\otimes E_d\\
&\quad+(q-q^{-1})[(\alpha_d|\alpha_k)]_q[\![E_d,E_j]\!]\sigma_dK_dK_i\otimes[\![E_d,E_i]\!]\\
&\quad+(q-q^{-1})[(\alpha_d|\alpha_j)]_qE_jK_d^2K_i\otimes\{q^{(\alpha_d|\alpha_j)}[\![E_d,E_i]\!]E_d+q^{(\alpha_d|\alpha_k)}E_d[\![E_d,E_i]\!]\}+K_d^2K_iK_j\otimes[\![[\![E_d,E_i]\!],[\![E_d,E_j]\!]]\!].
\end{aligned}$$
(3.16)

Furthermore, by Lemma 2.1, we have the following equality in $(\mathcal{U}_d'^{\geq 0}/\langle(2.6),(2.7)\rangle)^{\otimes 2}$:

$$
\begin{aligned}
&\Big\llbracket \Delta(\llbracket E_d,E_i\rrbracket),\Delta(\llbracket E_d,E_j\rrbracket)\Big\rrbracket,\Delta(\llbracket E_d,E_k\rrbracket)\Big\rrbracket \\
&= \llbracket\llbracket\llbracket E_d,E_i\rrbracket,\llbracket E_d,E_j\rrbracket\rrbracket,\llbracket E_d,E_k\rrbracket\rrbracket\otimes 1 \ \cdots \textcircled{1}\\
&\quad +(q-q^{-1})[(\alpha_d|\alpha_k)]_q\llbracket\llbracket\llbracket E_d,E_i\rrbracket,\llbracket E_d,E_j\rrbracket\rrbracket,E_k\rrbracket\sigma_dK_d\otimes E_d \ \cdots \textcircled{2}\\
&\quad -(q-q^{-1})[(\alpha_d|\alpha_k)]_q\Big\{q^{(\alpha_d|\alpha_k)}\llbracket E_i,\llbracket E_d,E_j\rrbracket\rrbracket\llbracket E_d,E_k\rrbracket+q^{-(\alpha_d|\alpha_k)}\llbracket E_d,E_k\rrbracket\llbracket E_i,\llbracket E_d,E_j\rrbracket\rrbracket\Big\}\sigma_dK_d\otimes E_d \ \cdots \textcircled{3}\\
&\quad +(q-q^{-1})[(\alpha_d|\alpha_k)]_q\big\{(q-q^{-1})[(\alpha_d|\alpha_j)]_q\llbracket E_d,E_j\rrbracket\llbracket E_d,E_k\rrbracket-q^{(\alpha_d|\alpha_j)}\big\llbracket\llbracket E_d,E_j\rrbracket,\llbracket E_d,E_k\rrbracket\big\rrbracket\big\}\sigma_dK_dK_i\otimes\llbracket E_d,E_i\rrbracket \ \cdots \textcircled{4}\\
&\quad +(q-q^{-1})^2[(\alpha_d|\alpha_k)]_q^2\Big[\{(q-q^{-1})[(\alpha_d|\alpha_k)]_q\llbracket E_d,E_j\rrbracket E_k+q^{-(\alpha_d|\alpha_k)}\llbracket\llbracket E_d,E_j\rrbracket,E_k\rrbracket\}K_d^2K_i\otimes\llbracket E_d,E_i\rrbracket E_d\\
&\qquad\qquad\qquad +q^{-2(\alpha_d|\alpha_k)}E_k\llbracket E_d,E_j\rrbracket K_d^2K_i\otimes\llbracket\llbracket E_d,E_i\rrbracket,E_d\rrbracket\Big] \ \cdots \textcircled{5}\\
&\quad +(q-q^{-1})[(\alpha_d|\alpha_k)]_q\llbracket E_d,E_j\rrbracket K_d^2K_iK_k\otimes\big\llbracket\llbracket E_d,E_i\rrbracket,\llbracket E_d,E_k\rrbracket\big\rrbracket \ \cdots \textcircled{6}\\
&\quad +(q-q^{-1})[(\alpha_d|\alpha_j)]_q\{(q-q^{-1})[(\alpha_d|2\alpha_k+\alpha_i)]_qE_j\llbracket E_d,E_k\rrbracket+q^{-(\alpha_d|2\alpha_k+\alpha_i)}\big\llbracket E_j,\llbracket E_d,E_k\rrbracket\big\rrbracket\}K_d^2K_1\\
&\qquad\qquad\qquad \otimes\{q^{(\alpha_d|\alpha_j)}\llbracket E_d,E_i\rrbracket E_d+q^{(\alpha_d|\alpha_k)}E_d\llbracket E_d,E_i\rrbracket\} \ \cdots \textcircled{7}\\
&\quad +(q-q^{-1})[(\alpha_d|\alpha_j)]_qE_j\sigma_dK_d^3K_iK_k\otimes\big\llbracket\{q^{(\alpha_d|\alpha_j)}\llbracket E_d,E_i\rrbracket E_d+q^{(\alpha_d|\alpha_k)}E_d\llbracket E_d,E_i\rrbracket\},\llbracket E_d,E_k\rrbracket\big\rrbracket \ \cdots \textcircled{8}\\
&\quad +(q-q^{-1})[(\alpha_d|\alpha_k)]_q\llbracket E_d,E_k\rrbracket K_d^2K_iK_j\otimes\big\llbracket\llbracket E_d,E_i\rrbracket,\llbracket E_d,E_j\rrbracket\big\rrbracket \ \cdots \textcircled{9}\\
&\quad +(q-q^{-1})[(\alpha_d|\alpha_k)]_qE_k\sigma_dK_d^3K_iK_j\\
&\qquad\qquad \otimes\Big\{(q-q^{-1})[2(\alpha_d|\alpha_k)]_q\big\llbracket\llbracket E_d,E_i\rrbracket,\llbracket E_d,E_j\rrbracket\big\rrbracket E_d+q^{-2(\alpha_d|\alpha_k)}\Big\llbracket\big\llbracket\llbracket E_d,E_i\rrbracket,\llbracket E_d,E_j\rrbracket\big\rrbracket,E_d\Big\rrbracket\Big\} \ \cdots \textcircled{10}\\
&\quad +\sigma_dK_d^3K_iK_jK_k\otimes\llbracket\llbracket\llbracket E_d,E_i\rrbracket,\llbracket E_d,E_j\rrbracket\rrbracket,\llbracket E_d,E_k\rrbracket\rrbracket \ \cdots \textcircled{11}. \qquad (3.17)
\end{aligned}
$$

We will give detailed proofs of (3.16) and (3.17) in section 4. In the following, we denote by $\textcircled{a}_{jk}$ the term $\textcircled{a}$ above for each $a=1,\ldots,11$. By Lemma 2.4, we have the following equality in $(\mathcal{U}_d'^{\geq 0}/\langle(2.6),(2.7)\rangle)^{\otimes 2}$:

$$
\begin{aligned}
&[(\alpha_d|\alpha_j)]_q(\textcircled{2}_{jk}+\textcircled{3}_{jk})=(q-q^{-1})[(\alpha_d|\alpha_j)]_q[(\alpha_d|\alpha_k)]_q\Big\{\Big\llbracket\big\llbracket\llbracket E_d,E_i\rrbracket,\llbracket\llbracket E_d,E_j\rrbracket,E_k\rrbracket\big\rrbracket\Big\rrbracket-q^{(\alpha_d|\alpha_j)}E_i\llbracket E_d,E_k\rrbracket\llbracket E_d,E_j\rrbracket\\
&\quad +q^{(\alpha_d|\alpha_k)}\llbracket E_d,E_j\rrbracket\llbracket E_d,E_k\rrbracket E_i-q^{(\alpha_d|\alpha_k)}E_i\llbracket E_d,E_j\rrbracket\llbracket E_d,E_k\rrbracket+q^{(\alpha_d|\alpha_j)}\llbracket E_d,E_k\rrbracket\llbracket E_d,E_j\rrbracket E_i\Big\}\sigma_dK_d\otimes E_d. \qquad (3.18)
\end{aligned}
$$

A detailed proof of (3.18) is also given in section 4. Exchanging j for k, we see that

$$
\begin{aligned}
&[(\alpha_d|\alpha_k)]_q(\textcircled{2}_{kj}+\textcircled{3}_{kj})=(q-q^{-1})[(\alpha_d|\alpha_k)]_q[(\alpha_d|\alpha_j)]_q\Big\{\Big\llbracket\big\llbracket\llbracket E_d,E_i\rrbracket,\llbracket\llbracket E_d,E_k\rrbracket,E_j\rrbracket\big\rrbracket\Big\rrbracket-q^{(\alpha_d|\alpha_k)}E_i\llbracket E_d,E_j\rrbracket\llbracket E_d,E_k\rrbracket\\
&\quad +q^{(\alpha_d|\alpha_j)}\llbracket E_d,E_k\rrbracket\llbracket E_d,E_j\rrbracket E_i-q^{(\alpha_d|\alpha_j)}E_i\llbracket E_d,E_k\rrbracket\llbracket E_d,E_j\rrbracket+q^{(\alpha_d|\alpha_k)}\llbracket E_d,E_j\rrbracket\llbracket E_d,E_k\rrbracket E_i\Big\}\sigma_dK_d\otimes E_d
\end{aligned}
$$

in $(\mathcal{U}_d'^{\geq 0}/\langle(2.6),(2.7)\rangle)^{\otimes 2}$. Thus we get the following equality in $(\mathcal{U}_d'^{\geq 0}/\langle(2.6),(2.7)\rangle)^{\otimes 2}$:

$$
-[(\alpha_d|\alpha_j)]_q(\textcircled{2}_{jk}+\textcircled{3}_{jk})+[(\alpha_d|\alpha_k)]_q(\textcircled{2}_{kj}+\textcircled{3}_{kj})=0. \qquad (3.19)
$$

We see that

$$
\begin{aligned}
[(\alpha_d|\alpha_j)]_q\,\textcircled{4}_{jk}&=(q-q^{-1})[(\alpha_d|\alpha_j)]_q[(\alpha_d|\alpha_k)]_q\big\{(q-q^{-1})[(\alpha_d|\alpha_j)]_q\llbracket E_d,E_j\rrbracket\llbracket E_d,E_k\rrbracket\\
&\qquad -q^{(\alpha_d|\alpha_j)}\big\llbracket\llbracket E_d,E_j\rrbracket,\llbracket E_d,E_k\rrbracket\big\rrbracket\big\}\sigma_dK_dK_i\otimes\llbracket E_d,E_i\rrbracket\\
&=(q-q^{-1})[(\alpha_d|\alpha_k)]_q[(\alpha_d|\alpha_j)]_q\{q^{-(\alpha_d|\alpha_j)}\llbracket E_d,E_j\rrbracket\llbracket E_d,E_k\rrbracket-q^{-(\alpha_d|\alpha_k)}\llbracket E_d,E_k\rrbracket\llbracket E_d,E_j\rrbracket\}\sigma_dK_dK_i\otimes\llbracket E_d,E_i\rrbracket,
\end{aligned}
$$

and hence that

$$
\begin{aligned}
[(\alpha_d|\alpha_k)]_q\,\textcircled{4}_{kj}&=(q-q^{-1})[(\alpha_d|\alpha_k)]_q[(\alpha_d|\alpha_j)]_q\big\{(q-q^{-1})[(\alpha_d|\alpha_k)]_q\llbracket E_d,E_k\rrbracket\llbracket E_d,E_j\rrbracket\\
&\qquad -q^{(\alpha_d|\alpha_k)}\big\llbracket\llbracket E_d,E_k\rrbracket,\llbracket E_d,E_j\rrbracket\big\rrbracket\big\}\sigma_dK_dK_i\otimes\llbracket E_d,E_i\rrbracket\\
&=(q-q^{-1})[(\alpha_d|\alpha_j)]_q[(\alpha_d|\alpha_k)]_q\{q^{-(\alpha_d|\alpha_k)}\llbracket E_d,E_k\rrbracket\llbracket E_d,E_j\rrbracket-q^{-(\alpha_d|\alpha_j)}\llbracket E_d,E_j\rrbracket\llbracket E_d,E_k\rrbracket\}\sigma_dK_dK_i\otimes\llbracket E_d,E_i\rrbracket.
\end{aligned}
$$

Thus we get

$$
-[(\alpha_d|\alpha_j)]_q\,\textcircled{4}_{jk}+[(\alpha_d|\alpha_k)]_q\,\textcircled{4}_{kj}=0. \qquad (3.20)
$$

By Lemma 2.3, we have the following equality in $(\mathcal{U}_d'^{\geq 0}/\langle(2.6),(2.7)\rangle)^{\otimes 2}$:

$$\begin{aligned}
⑤_{jk} &= (q-q^{-1})^2[(\alpha_d|\alpha_k)]_q^2\Big[-q^{(\alpha_d|\alpha_1)}\{q^{(\alpha_d|\alpha_k)}[\![E_d,E_j]\!]E_k - q^{-2(\alpha_d|\alpha_k)}E_k[\![E_d,E_j]\!]\} \\
&\qquad - q^{-2(\alpha_d|\alpha_k)}(q-q^{-1})[(\alpha_d|\alpha_i)]_qE_k[\![E_d,E_j]\!]\Big]K_d^2K_i\otimes E_d[\![E_d,E_i]\!] \\
&= (q-q^{-1})^2[(\alpha_d|\alpha_k)]_q^2\Big\{-q^{-(\alpha_d|\alpha_j)}[\![E_d,E_j]\!]E_k + q^{-(\alpha_d|\alpha_i+2\alpha_k)}E_k[\![E_d,E_j]\!]\Big\}K_d^2K_i\otimes E_d[\![E_d,E_i]\!] \\
&= (q-q^{-1})^2[(\alpha_d|\alpha_k)]_q^2\Big\{-q^{-(\alpha_d|\alpha_j)}[\![E_d,E_j]\!]E_k + q^{(\alpha_d|\alpha_j-\alpha_k)}E_k[\![E_d,E_j]\!]\Big\}K_d^2K_1\otimes E_d[\![E_d,E_i]\!],
\end{aligned}$$

$$\begin{aligned}
⑦_{jk} &= (q-q^{-1})^2[(\alpha_d|\alpha_j)]_q[(\alpha_d|\alpha_k)]_q\Big\{q^{(\alpha_d|2\alpha_k+\alpha_i)}E_j[\![E_d,E_k]\!] - q^{-(\alpha_d|2\alpha_k+\alpha_i+\alpha_j)}[\![E_d,E_k]\!]E_j\Big\}K_d^2K_1\otimes E_d[\![E_d,E_1]\!] \\
&= (q-q^{-1})^2[(\alpha_d|\alpha_j)]_q[(\alpha_d|\alpha_k)]_q\Big\{q^{(\alpha_d|\alpha_k-\alpha_j)}E_j[\![E_d,E_k]\!] - q^{-(\alpha_d|\alpha_k)}[\![E_d,E_k]\!]E_j\Big\}K_d^2K_1\otimes E_d[\![E_d,E_1]\!].
\end{aligned}$$

Thus we see that

$$\begin{aligned}
&[(\alpha_d|\alpha_j)]_q(⑤_{jk}+⑦_{jk}) \\
&= (q-q^{-1})^2[(\alpha_d|\alpha_j)]_q[(\alpha_d|\alpha_k)]_q\Big[[(\alpha_d|\alpha_k)]_q\{-q^{-(\alpha_d|\alpha_j)}[\![E_d,E_j]\!]E_k + q^{(\alpha_d|\alpha_j-\alpha_k)}E_k[\![E_d,E_j]\!]\} \\
&\qquad + [(\alpha_d|\alpha_j)]_q\{q^{(\alpha_d|\alpha_k-\alpha_j)}E_j[\![E_d,E_k]\!] - q^{-(\alpha_d|\alpha_k)}[\![E_d,E_k]\!]E_j\}\Big]K_d^2K_1\otimes E_d[\![E_d,E_1]\!],
\end{aligned}$$

and hence that

$$\begin{aligned}
&[(\alpha_d|\alpha_k)]_q(⑤_{kj}+⑦_{kj}) \\
&= (q-q^{-1})^2[(\alpha_d|\alpha_k)]_q[(\alpha_d|\alpha_j)]_q\Big[[(\alpha_d|\alpha_j)]_q\{-q^{-(\alpha_d|\alpha_k)}[\![E_d,E_k]\!]E_j + q^{(\alpha_d|\alpha_k-\alpha_j)}E_j[\![E_d,E_k]\!]\} \\
&\qquad + [(\alpha_d|\alpha_k)]_q\{q^{(\alpha_d|\alpha_j-\alpha_k)}E_k[\![E_d,E_j]\!] - q^{-(\alpha_d|\alpha_j)}[\![E_d,E_j]\!]E_k\}\Big]K_d^2K_1\otimes E_d[\![E_d,E_1]\!].
\end{aligned}$$

Therefore we get

$$-[(\alpha_d|\alpha_j)]_q(⑤_{jk}+⑦_{jk}) + [(\alpha_d|\alpha_k)]_q(⑤_{kj}+⑦_{kj}) = 0. \tag{3.21}$$

It is clear that

$$-[(\alpha_d|\alpha_j)]_q\,⑥_{jk} + [(\alpha_d|\alpha_k)]_q\,⑨_{kj} = 0. \tag{3.22}$$

By Lemma 2.4, we have the following equality in $(\mathcal{U}_d'^{\geq 0}/\langle(2.6),(2.7)\rangle)^{\otimes 2}$:

$$\begin{aligned}
&[(\alpha_d|\alpha_j)]_q\,⑧_{jk} = (q-q^{-1})^2[(\alpha_d|\alpha_j)]_q^2[(\alpha_d|\alpha_k)]_qE_j\sigma_dK_d^3K_iK_k\otimes[\![E_d[\![E_d,E_i]\!],[\![E_d,E_k]\!]]\!] \\
&= (q-q^{-1})^2[(\alpha_d|\alpha_j)]_q^2[(\alpha_d|\alpha_k)]_qE_j\sigma_dK_d^3K_iK_k\otimes E_d\big[\!\big[[\![E_d,E_i]\!],[\![E_d,E_k]\!]\big]\!\big], \\
&[(\alpha_d|\alpha_k)]_q\,⑩_{kj} = (q-q^{-1})[(\alpha_d|\alpha_j)]_q[(\alpha_d|\alpha_k)]_qE_j\sigma_dK_d^3K_iK_k\otimes\Big\{(q-q^{-1})[2(\alpha_d|\alpha_j)]_q\big[\!\big[[\![E_d,E_1]\!],[\![E_d,E_k]\!]\big]\!\big]E_d \\
&\qquad + q^{-2(\alpha_d|\alpha_j)}\{\big[\!\big[[\![E_d,E_i]\!],[\![E_d,E_k]\!]\big]\!\big]E_d - (-1)^{2\cdot 1}q^{-(\alpha_d|\alpha_i+\alpha_k)}E_d\big[\!\big[[\![E_d,E_i]\!],[\![E_d,E_k]\!]\big]\!\big]\}\Big\} \\
&= (q-q^{-1})[(\alpha_d|\alpha_j)]_q[(\alpha_d|\alpha_k)]_qE_j\sigma_dK_d^3K_iK_k\otimes\Big\{q^{2(\alpha_d|\alpha_j)}\big[\!\big[[\![E_d,E_i]\!],[\![E_d,E_k]\!]\big]\!\big]E_d - q^{-(\alpha_d|\alpha_j)}E_d\big[\!\big[[\![E_d,E_i]\!],[\![E_d,E_k]\!]\big]\!\big]\Big\} \\
&= (q-q^{-1})^2[(\alpha_d|\alpha_j)]_q^2[(\alpha_d|\alpha_k)]_qE_j\sigma_dK_d^3K_iK_k\otimes E_d\big[\!\big[[\![E_d,E_i]\!],[\![E_d,E_k]\!]\big]\!\big].
\end{aligned}$$

Thus we get

$$-[(\alpha_d|\alpha_j)]_q\,⑧_{jk} + [(\alpha_d|\alpha_k)]_q\,⑩_{kj} = 0. \tag{3.23}$$

Thanks to (3.19)–(3.23), we get the equality (3.14) in $(\mathcal{U}_d'^{\geq 0}/\langle(2.6),(2.7)\rangle)^{\otimes 2}$.

Lemma 3.3. *Let X_λ be an element of $\mathcal{U}_{d,\lambda}'^{>0}$ with $\lambda\in Q_d^+$. Let us write $\Delta(X_\mu)$ as follows:*

$$\Delta(X_\lambda) = \sum_{\mu,\nu\in Q_d^+} X_\mu\sigma_d^{p(\nu)}K_\nu\otimes X_\nu \tag{3.24}$$

with $X_\mu\in\mathcal{U}_{d,\mu}'^{>0}$ and $X_\nu\in\mathcal{U}_{d,\nu}'^{>0}$, where the sum is over all $\mu,\nu\in Q_d^+$ with $\lambda=\mu+\nu$. Then the following equality holds:

$$\Delta(\Psi_d^{-1}(X_\lambda)) = \sum_{\mu,\nu\in Q_d^+} \sigma_d^{p(\mu)}\Psi_d^{-1}(X_\nu)\otimes\Psi_d^{-1}(X_\mu)K_\nu^{-1}. \tag{3.25}$$

Proof. We use the induction on the height of λ. In the case where $\lambda \in \Pi_d \amalg \{0\}$, the claim is clear. We suppose that the claim is valid for some $\lambda \in Q_d^+$. Put $\lambda' = \alpha + \lambda$ with $\alpha \in \Pi_d$. Then we see that

$$\begin{aligned}\Delta(E_\alpha X_\lambda) &= (E_\alpha \otimes 1 + \sigma_d^{p(\alpha)} K_\alpha \otimes E_\alpha)(\sum_{\mu,\nu} X_\mu \sigma_d^{p(\nu)} K_\nu \otimes X_\nu) \\ &= \sum_{\mu,\nu} E_\alpha X_\mu \sigma_d^{p(\nu)} K_\nu \otimes X_\nu + \sum_{\mu,\nu} \sigma_d^{p(\alpha)} K_\alpha X_\mu \sigma_d^{p(\nu)} K_\nu \otimes E_\alpha X_\nu \\ &= \sum_{\mu,\nu} E_\alpha X_\mu \sigma_d^{p(\nu)} K_\nu \otimes X_\nu + \sum_{\mu,\nu} (-1)^{p(\alpha)p(\mu)} q^{(\alpha|\mu)} X_\mu \sigma_d^{p(\alpha+\nu)} K_{\alpha+\nu} \otimes E_\alpha X_\nu\end{aligned}$$

and that

$$\begin{aligned}&\Delta(\Psi_d^{-1}(E_\alpha X_\lambda)) = (F_\alpha \otimes K_\alpha^{-1} + \sigma_d^{p(\alpha)} \otimes F_\alpha)(\sum_{\mu,\nu} \sigma_d^{p(\mu)} \Psi_d^{-1}(X_\nu) \otimes \Psi_d^{-1}(X_\mu) K_\nu^{-1}) \\ &= \sum_{\mu,\nu} F_\alpha \sigma_d^{p(\mu)} \Psi_d^{-1}(X_\nu) \otimes K_\alpha^{-1} \Psi_d^{-1}(X_\mu) K_\nu^{-1} + \sum_{\mu,\nu} \sigma_d^{p(\mu)} \sigma_d^{p(\alpha)} \Psi_d^{-1}(X_\nu) \otimes F_\alpha \Psi_d^{-1}(X_\mu) K_\nu^{-1} \\ &= \sum_{\mu,\nu} \sigma_d^{p(\mu)} \Psi_d^{-1}(E_\alpha X_\nu) \otimes (-1)^{p(\alpha)p(\mu)} q^{(\alpha|\mu)} \Psi_d^{-1}(X_\mu) K_{\alpha+\nu}^{-1} + \sum_{\mu,\nu} \sigma_d^{p(\alpha+\mu)} \Psi_d^{-1}(X_\nu) \otimes \Psi_d^{-1}(E_\alpha X_\mu) K_\nu^{-1}.\end{aligned}$$

Thus the claim is also valid for λ'. □

Proposition 3.4. (1) *Let (Δ, ε, S) be the Hopf algebra structure on $\mathcal{U}_d'$ introduced in Proposition 3.1. Let $\mathcal{SR} \in \mathcal{U}_d'^{>0}$ be an arbitrary element displayed in (2.6)–(2.9), and set $\mathcal{SR}^- := \Psi_d(\mathcal{SR})$. Then the following equalities hold:*

$$\Delta(\mathcal{SR}) = \mathcal{SR} \otimes 1 + \sigma_d^{p(\mathrm{wt}(\mathcal{SR}))} K_{\mathrm{wt}(\mathcal{SR})} \otimes \mathcal{SR}, \qquad \Delta(\mathcal{SR}^-) = \mathcal{SR}^- \otimes K_{\mathrm{wt}(\mathcal{SR})}^{-1} + \sigma_d^{p(\mathrm{wt}(\mathcal{SR}))} \otimes \mathcal{SR}^-, \quad (3.26)$$

$$S(\mathcal{SR}) = -\sigma_d^{\mathrm{wt}(\mathcal{SR})} K_{\mathrm{wt}(\mathcal{SR})}^{-1} \mathcal{SR}, \qquad S(\mathcal{SR}^-) = -(-1)^{\mathrm{wt}(\mathcal{SR})} \mathcal{SR}^- \sigma_d^{\mathrm{wt}(\mathcal{SR})} K_{\mathrm{wt}(\mathcal{SR})}. \quad (3.27)$$

(2) *Let $\mathcal{L}$ be the two-sided ideal of $\mathcal{U}_d'^{\geq 0}$ generated by the elements displayed in (2.6) and (2.7). If $\mathcal{SR} \in \mathcal{U}_d'^{>0}$ is an arbitrary element displayed in (2.10), then the left (resp. right) equality of (3.26) holds modulo $\mathcal{L} \otimes \mathcal{U}_d'^{>0} + \mathcal{U}_d'^{\geq 0} \otimes \mathcal{L}$ (resp. $\Psi_d(\mathcal{L}) \otimes \mathcal{U}_d'^{\leq 0} + \mathcal{U}_d'^{\leq 0} \otimes \Psi_d(\mathcal{L})$), and the left (resp. right) equality of (3.27) holds modulo $\mathcal{L}$ (resp. $\Psi_d(\mathcal{L})$).*

Proof. (1) The left equality of (3.26) has been already proved by (3.8)(3.10)(3.11)(3.12).

Let us prove the right equality in (3.26). Note that $\Psi_d^{-1}(X) = \sigma_d \Psi_d(X) \sigma_d$ for all $X \in \mathcal{U}_d'$. Hence $\mathcal{SR}^- = \Psi_d(\mathcal{SR}) = \sigma_d \Psi_d^{-1}(\mathcal{SR}) \sigma_d$. By Lemma 3.3 and the left equality in (3.26), we see that

$$\begin{aligned}\Delta(\mathcal{SR}^-) &= \Delta(\sigma_d \Psi_d^{-1}(\mathcal{SR}) \sigma_d) = (\sigma_d \otimes \sigma_d) \Delta(\Psi_d^{-1}(\mathcal{SR}))(\sigma_d \otimes \sigma_d) \\ &= (\sigma_d \otimes \sigma_d)\big(\sigma_d^{p(\mathrm{wt}(\mathcal{SR}))} \otimes \Psi_d^{-1}(\mathcal{SR}) + \Psi_d^{-1}(\mathcal{SR}) \otimes K_{\mathrm{wt}(\mathcal{SR})}^{-1}\big)(\sigma_d \otimes \sigma_d) \\ &= \mathcal{SR}^- \otimes K_{\mathrm{wt}(\mathcal{SR})}^{-1} + \sigma_d^{p(\mathrm{wt}(\mathcal{SR}))} \otimes \mathcal{SR}^-.\end{aligned}$$

Let us prove the equalities of (3.27). By Proposition 3.1 and (3.26), we see that

$$\begin{aligned}0 &= \iota \circ \varepsilon(\mathcal{SR}) = m \circ (S \otimes id) \circ \Delta(\mathcal{SR}) = S(\mathcal{SR}) + \sigma_d^{-\mathrm{wt}(\mathcal{SR})} K_{\mathrm{wt}(\mathcal{SR})}^{-1} \mathcal{SR}, \\ 0 &= \iota \circ \varepsilon(\mathcal{SR}^-) = m \circ (S \otimes id) \circ \Delta(\mathcal{SR}^-) = S(\mathcal{SR}^-) K_{\mathrm{wt}(\mathcal{SR})}^{-1} + \sigma_d^{-\mathrm{wt}(\mathcal{SR})} \mathcal{SR}^-.\end{aligned}$$

Thus we get the equalities of (3.27).

(2) By (3.14), we see that the left equality of (3.26) holds modulo $\mathcal{L} \otimes \mathcal{U}_d'^{>0} + \mathcal{U}_d'^{\geq 0} \otimes \mathcal{L}$. As the above argument, by Lemma 3.3, we see that the right equality of (3.26) holds modulo $\Psi_d(\mathcal{L}) \otimes \mathcal{U}_d'^{\leq 0} + \mathcal{U}_d'^{\leq 0} \otimes \Psi_d(\mathcal{L})$. By (1), we have both $S(\mathcal{L}) \subset \mathcal{L}$ and $S(\Psi_d(\mathcal{L})) \subset \Psi_d(\mathcal{L})$. Thus, as the above argument, we see that the left (resp. right) equality of (3.27) holds modulo $\mathcal{L}$ (resp. $\Psi_d(\mathcal{L})$). □

Theorem 3.5 ([1]). *For each $d \in \mathcal{D}$, there is a unique Hopf algebra structure (Δ, ε, S) on U_d' satisfying the same formulas as in Proposition 3.1.*

Proof. Let $\mathcal{I}$ be the two-sided ideal of $\mathcal{U}_d'$ generated by the elements (2.6)–(2.11). Then $U_d' = \mathcal{U}_d'/\mathcal{I}$. By Proposition 3.4, we see that $\Delta(\mathcal{I}) \subset \mathcal{I} \otimes \mathcal{U}_d' + \mathcal{U}_d' \otimes \mathcal{I}$ and $S(\mathcal{I}) \subset \mathcal{I}$. In addition, we note that $(\mathcal{U}_d' \otimes \mathcal{U}_d')/(\mathcal{I} \otimes \mathcal{U}_d' + \mathcal{U}_d' \otimes \mathcal{I}) \simeq U_d' \otimes U_d'$. Thus, by Proposition 3.1, we see that the Hopf algebra structure (Δ, ε, S) on U_d' is induced by the Hopf algebra structure (Δ, ε, S) on $\mathcal{U}_d'$. □

4 Appendix

In this section, we will give detailed proofs of the equalities (3.16), (3.17), and (3.18).

Let $d \neq 4$ and $\{i,j,k,d\} = \mathrm{I}$ with $i<j<k$. We put $E_l = E_{l,d}$, $K_l = K_{l,d}$, and $\alpha_l = \alpha_{l,d}$ for each $l \in \mathrm{I}$. Note that $p(\alpha_d)=1$, $p(\alpha_i)=p(\alpha_j)=p(\alpha_k)=0$ and $(\alpha_d|\alpha_d)=(\alpha_d|\alpha_i+\alpha_j+\alpha_k)=0$.

Let us prove (3.16). By Lemma 2.1, we see that

$$
\begin{aligned}
&\big[\!\!\big[\Delta([\![E_d,E_i]\!]),\Delta([\![E_d,E_j]\!])\big]\!\!\big]\\
&=\big[\!\!\big[[\![E_d,E_i]\!]\otimes 1+(q-q^{-1})[(\alpha_d|\alpha_i)]_q E_i\sigma_dK_d\otimes E_d+\sigma_dK_dK_i\otimes[\![E_d,E_i]\!]\,,\\
&\qquad [\![E_d,E_j]\!]\otimes 1+(q-q^{-1})[(\alpha_d|\alpha_j)]_q E_j\sigma_dK_d\otimes E_d+\sigma_dK_dK_j\otimes[\![E_d,E_j]\!]\big]\!\!\big]\\
&=[\![[\![E_d,E_i]\!],[\![E_d,E_j]\!]]\!]\otimes 1+(q-q^{-1})[(\alpha_d|\alpha_j)]_q[\![[\![E_d,E_i]\!],E_j]\!]\sigma_dK_d\otimes E_d+0\\
&\qquad +(q-q^{-1})[(\alpha_d|\alpha_i)]_q\{-q^{(\alpha_d|\alpha_j)}E_i[\![E_d,E_j]\!]+q^{-(\alpha_d|\alpha_i+\alpha_j)}[\![E_d,E_j]\!]E_i\}\sigma_dK_d\otimes E_d\\
&\qquad +0\quad (\because E_d^2=E_{d,d}^2=0)\quad +0\quad (\because E_d[\![E_d,E_j]\!]+q^{-(\alpha_d|\alpha_j)}[\![E_d,E_j]\!]E_d=[\![E_d,[\![E_d,E_j]\!]]\!]=0)\\
&\qquad -(q-q^{-1})[(\alpha_d|\alpha_i+\alpha_j)]_q[\![E_d,E_j]\!]\sigma_dK_dK_i\otimes[\![E_d,E_i]\!]\\
&\quad +(q-q^{-1})[(\alpha_d|\alpha_j)]_qE_jK_d^2K_i\otimes\{q^{(\alpha_d|\alpha_j)}[\![E_d,E_i]\!]E_d+q^{-(\alpha_d|\alpha_i+\alpha_j)}E_d[\![E_d,E_i]\!]\}+K_d^2K_iK_j\otimes[\![[\![E_d,E_i]\!],[\![E_d,E_j]\!]]\!]\\
&=[\![[\![E_d,E_i]\!],[\![E_d,E_j]\!]]\!]\otimes 1+(q^{(\alpha_d|\alpha_j)}-q^{-(\alpha_d|\alpha_j)})[\![[\![E_d,E_j]\!],E_i]\!]\sigma_dK_d\otimes E_d\\
&\qquad +(q^{(\alpha_d|\alpha_i)}-q^{-(\alpha_d|\alpha_i)})\{-q^{(\alpha_d|\alpha_j)}E_i[\![E_d,E_j]\!]+q^{(\alpha_d|\alpha_k)}[\![E_d,E_j]\!]E_i\}\sigma_dK_d\otimes E_d\\
&\qquad +(q-q^{-1})[(\alpha_d|\alpha_k)]_q[\![E_d,E_j]\!]\sigma_dK_dK_i\otimes[\![E_d,E_i]\!]\\
&\quad +(q-q^{-1})[(\alpha_d|\alpha_j)]_qE_jK_d^2K_i\otimes\{q^{(\alpha_d|\alpha_j)}[\![E_d,E_i]\!]E_d+q^{(\alpha_d|\alpha_k)}E_d[\![E_d,E_i]\!]\}+K_d^2K_iK_j\otimes[\![[\![E_d,E_i]\!],[\![E_d,E_j]\!]]\!]\\
&=[\![[\![E_d,E_i]\!],[\![E_d,E_j]\!]]\!]\otimes 1+(q^{(\alpha_d|\alpha_j)}-q^{-(\alpha_d|\alpha_j)})([\![E_d,E_j]\!]E_i-q^{-(\alpha_d|\alpha_i)}E_i[\![E_d,E_j]\!])\sigma_dK_d\otimes E_d\\
&\qquad +(q^{(\alpha_d|\alpha_i)}-q^{-(\alpha_d|\alpha_i)})\{-q^{(\alpha_d|\alpha_j)}E_i[\![E_d,E_j]\!]+q^{(\alpha_d|\alpha_k)}[\![E_d,E_j]\!]E_i\}\sigma_dK_d\otimes E_d\\
&\qquad +(q-q^{-1})[(\alpha_d|\alpha_k)]_q[\![E_d,E_j]\!]\sigma_dK_dK_i\otimes[\![E_d,E_i]\!]\\
&\quad +(q-q^{-1})[(\alpha_d|\alpha_j)]_qE_jK_d^2K_i\otimes\{q^{(\alpha_d|\alpha_j)}[\![E_d,E_i]\!]E_d+q^{(\alpha_d|\alpha_k)}E_d[\![E_d,E_i]\!]\}+K_d^2K_iK_j\otimes[\![[\![E_d,E_i]\!],[\![E_d,E_j]\!]]\!].
\end{aligned}
$$

Here, by combining the second term with the third term, we see that

$$
\begin{aligned}
&\big[\!\!\big[\Delta([\![E_d,E_i]\!]),\Delta([\![E_d,E_j]\!])\big]\!\!\big]\\
&=[\![[\![E_d,E_i]\!],[\![E_d,E_j]\!]]\!]\otimes 1+\big\{(q^{(\alpha_d|\alpha_j)}-q^{(\alpha_d|\alpha_k-\alpha_i)})[\![E_d,E_j]\!]E_i+(q-q^{-1})[(\alpha_d|\alpha_k)]_qE_i[\![E_d,E_j]\!]\big\}\sigma_dK_d\otimes E_d\\
&\qquad +(q-q^{-1})[(\alpha_d|\alpha_k)]_q[\![E_d,E_j]\!]\sigma_dK_dK_i\otimes[\![E_d,E_i]\!]\\
&\quad +(q-q^{-1})[(\alpha_d|\alpha_j)]_qE_jK_d^2K_i\otimes\{q^{(\alpha_d|\alpha_j)}[\![E_d,E_i]\!]E_d+q^{(\alpha_d|\alpha_k)}E_d[\![E_d,E_i]\!]\}+K_d^2K_iK_j\otimes[\![[\![E_d,E_i]\!],[\![E_d,E_j]\!]]\!]\\
&=[\![[\![E_d,E_i]\!],[\![E_d,E_j]\!]]\!]\otimes 1+(q-q^{-1})[(\alpha_d|\alpha_k)]_q\big\{-q^{-(\alpha_d|\alpha_i)}[\![E_d,E_j]\!]E_i+E_i[\![E_d,E_j]\!]\big\}\sigma_dK_d\otimes E_d\\
&\qquad +(q-q^{-1})[(\alpha_d|\alpha_k)]_q[\![E_d,E_j]\!]\sigma_dK_dK_i\otimes[\![E_d,E_i]\!]\\
&\quad +(q-q^{-1})[(\alpha_d|\alpha_j)]_qE_jK_d^2K_i\otimes\{q^{(\alpha_d|\alpha_j)}[\![E_d,E_i]\!]E_d+q^{(\alpha_d|\alpha_k)}E_d[\![E_d,E_i]\!]\}+K_d^2K_iK_j\otimes[\![[\![E_d,E_i]\!],[\![E_d,E_j]\!]]\!]\\
&=[\![[\![E_d,E_i]\!],[\![E_d,E_j]\!]]\!]\otimes 1+(q-q^{-1})[(\alpha_d|\alpha_k)]_q\big[\!\!\big[E_i,[\![E_d,E_j]\!]\big]\!\!\big]\sigma_dK_d\otimes E_d\\
&\qquad +(q-q^{-1})[(\alpha_d|\alpha_k)]_q[\![E_d,E_j]\!]\sigma_dK_dK_i\otimes[\![E_d,E_i]\!]\\
&\quad +(q-q^{-1})[(\alpha_d|\alpha_j)]_qE_jK_d^2K_i\otimes\{q^{(\alpha_d|\alpha_j)}[\![E_d,E_i]\!]E_d+q^{(\alpha_d|\alpha_k)}E_d[\![E_d,E_i]\!]\}+K_d^2K_iK_j\otimes[\![[\![E_d,E_i]\!],[\![E_d,E_j]\!]]\!].
\end{aligned}
$$

Let us prove (3.17). By Lemma 2.1, we see that

$$
\begin{aligned}
&\Big[\!\!\Big[\big[\!\!\big[\Delta([\![E_d,E_i]\!]),\Delta([\![E_d,E_j]\!])\big]\!\!\big],\Delta([\![E_d,E_k]\!])\Big]\!\!\Big]\\
&=\Big[\!\!\Big[[\![[\![E_d,E_i]\!],[\![E_d,E_j]\!]]\!]\otimes 1+(q-q^{-1})[(\alpha_d|\alpha_k)]_q\big[\!\!\big[E_i,[\![E_d,E_j]\!]\big]\!\!\big]\sigma_dK_d\otimes E_d\\
&\qquad +(q-q^{-1})[(\alpha_d|\alpha_k)]_q[\![E_d,E_j]\!]\sigma_dK_dK_i\otimes[\![E_d,E_i]\!]\\
&\quad +(q-q^{-1})[(\alpha_d|\alpha_j)]_qE_jK_d^2K_i\otimes\{q^{(\alpha_d|\alpha_j)}[\![E_d,E_i]\!]E_d+q^{(\alpha_d|\alpha_k)}E_d[\![E_d,E_i]\!]\}+K_d^2K_iK_j\otimes[\![[\![E_d,E_i]\!],[\![E_d,E_j]\!]]\!]\,,\\
&\qquad [\![E_d,E_k]\!]\otimes 1+(q-q^{-1})[(\alpha_d|\alpha_k)]_qE_k\sigma_dK_d\otimes E_d+\sigma_dK_dK_k\otimes[\![E_d,E_k]\!]\Big]\!\!\Big]\\
&=[\![[\![[\![E_d,E_i]\!],[\![E_d,E_j]\!]]\!],[\![E_d,E_k]\!]]\!]\otimes 1+(q-q^{-1})[(\alpha_d|\alpha_k)]_q[\![[\![[\![E_d,E_i]\!],[\![E_d,E_j]\!]]\!],E_k]\!]\sigma_dK_d\otimes E_d\quad +0
\end{aligned}
$$

$$
\begin{aligned}
&-(q-q^{-1})[(\alpha_d|\alpha_k)]_q\Big\{(q-q^{-1})[(\alpha_d|\alpha_k)]_q\llbracket E_i,\llbracket E_d,E_j\rrbracket\rrbracket\llbracket E_d,E_k\rrbracket \\
&\qquad\qquad +q^{-(\alpha_d|\alpha_k)}\Big\llbracket\llbracket E_i,\llbracket E_d,E_j\rrbracket\rrbracket,\llbracket E_d,E_k\rrbracket\Big\rrbracket\Big\}\sigma_d K_d\otimes E_d \\
&\quad +0\ \ +0\quad (\because \llbracket E_d,\llbracket E_d,E_k\rrbracket\rrbracket=0) \\
&+(q-q^{-1})[(\alpha_d|\alpha_k)]_q\Big\{(q-q^{-1})[(\alpha_d|\alpha_j)]_q\llbracket E_d,E_j\rrbracket\llbracket E_d,E_k\rrbracket \\
&\qquad\qquad +q^{(\alpha_d|\alpha_j)}\Big\llbracket\llbracket E_d,E_j\rrbracket,\llbracket E_d,E_k\rrbracket\Big\rrbracket\Big\}\sigma_d K_d K_i\otimes\llbracket E_d,E_i\rrbracket \\
&\quad +(q-q^{-1})^2[(\alpha_d|\alpha_k)]_q^2\Big[\{(q-q^{-1})[(\alpha_d|\alpha_k)]_q\llbracket E_d,E_j\rrbracket E_k+q^{-(\alpha_d|\alpha_k)}\llbracket\llbracket E_d,E_j\rrbracket,E_k\rrbracket\}K_d^2K_1\otimes\llbracket E_d,E_1\rrbracket E_d \\
&\qquad\qquad +q^{-2(\alpha_d|\alpha_k)}E_k\llbracket E_d,E_j\rrbracket K_d^2K_i\otimes\llbracket\llbracket E_d,E_i\rrbracket,E_d\rrbracket\Big] \\
&+(q-q^{-1})[(\alpha_d|\alpha_k)]_q\llbracket E_d,E_j\rrbracket K_d^2K_iK_k\otimes\Big\llbracket\llbracket E_d,E_i\rrbracket,\llbracket E_d,E_k\rrbracket\Big\rrbracket \\
&+(q-q^{-1})[(\alpha_d|\alpha_j)]_q\Big\{(q-q^{-1})[(\alpha_d|2\alpha_k+\alpha_i)]_qE_j\llbracket E_d,E_k\rrbracket+q^{-(\alpha_d|2\alpha_k+\alpha_i)}\Big\llbracket E_j,\llbracket E_d,E_k\rrbracket\Big\rrbracket\Big\}K_d^2K_i \\
&\qquad\qquad \otimes\Big\{q^{(\alpha_d|\alpha_j)}\llbracket E_d,E_1\rrbracket E_d+q^{(\alpha_d|\alpha_k)}E_d\llbracket E_d,E_i\rrbracket\Big\} \\
&\quad +0\quad (\because \text{Lemma 2.3},\ E_d^2=0) \\
&+(q-q^{-1})[(\alpha_d|\alpha_j)]_qE_j\sigma_dK_d^3K_iK_k\otimes\llbracket\{q^{(\alpha_d|\alpha_j)}\llbracket E_d,E_i\rrbracket E_d+q^{(\alpha_d|\alpha_k)}E_d\llbracket E_d,E_i\rrbracket\},\llbracket E_d,E_k\rrbracket\rrbracket \\
&+(q-q^{-1})[(\alpha_d|\alpha_k)]_q\llbracket E_d,E_k\rrbracket K_d^2K_iK_j\otimes\Big\llbracket\llbracket E_d,E_i\rrbracket,\llbracket E_d,E_j\rrbracket\Big\rrbracket \\
&+(q-q^{-1})[(\alpha_d|\alpha_k)]_qE_k\sigma_dK_d^3K_iK_j \\
&\qquad\qquad \otimes\Big\{(q-q^{-1})[2(\alpha_d|\alpha_k)]_q\Big\llbracket\llbracket E_d,E_i\rrbracket,\llbracket E_d,E_j\rrbracket\Big\rrbracket E_d+q^{-2(\alpha_d|\alpha_k)}\Big\llbracket\Big\llbracket\llbracket E_d,E_i\rrbracket,\llbracket E_d,E_j\rrbracket\Big\rrbracket,E_d\Big\rrbracket\Big\} \\
&+\sigma_dK_d^3K_iK_jK_k\otimes\llbracket\llbracket\llbracket E_d,E_i\rrbracket,\llbracket E_d,E_j\rrbracket\rrbracket,\llbracket E_d,E_k\rrbracket\rrbracket.
\end{aligned}
$$

Let us prove (3.18). By Lemma 2.4, we see that

$$
\begin{aligned}
&[(\alpha_d|\alpha_j)]_q(②_{jk}+③_{jk})=(q-q^{-1})[(\alpha_d|\alpha_j)]_q[(\alpha_d|\alpha_k)]_q\Big\{\Big\llbracket\Big\llbracket\llbracket E_d,E_i\rrbracket,\llbracket E_d,E_j\rrbracket\Big\rrbracket,E_k\Big\rrbracket \\
&\qquad -q^{(\alpha_d|\alpha_k)}\Big\llbracket E_i,\llbracket E_d,E_j\rrbracket\Big\rrbracket\llbracket E_d,E_k\rrbracket-q^{-(\alpha_d|\alpha_k)}\llbracket E_d,E_k\rrbracket\Big\llbracket E_i,\llbracket E_d,E_j\rrbracket\Big\rrbracket\Big\}\sigma_dK_d\otimes E_d \\
&=(q-q^{-1})[(\alpha_d|\alpha_j)]_q[(\alpha_d|\alpha_k)]_q\Big\{\Big\llbracket\Big\llbracket\llbracket E_d,E_i\rrbracket,\llbracket E_d,E_j\rrbracket\Big\rrbracket,E_k\Big\rrbracket \\
&\qquad -q^{(\alpha_d|\alpha_k)}(E_i\llbracket E_d,E_j\rrbracket-q^{-(\alpha_d|\alpha_i)}\llbracket E_d,E_j\rrbracket E_i)\llbracket E_d,E_k\rrbracket \\
&\qquad -q^{-(\alpha_d|\alpha_k)}\llbracket E_d,E_k\rrbracket(E_i\llbracket E_d,E_j\rrbracket-q^{-(\alpha_d|\alpha_i)}\llbracket E_d,E_j\rrbracket E_i)\Big\}\sigma_dK_d\otimes E_d \\
&=(q-q^{-1})[(\alpha_d|\alpha_j)]_q[(\alpha_d|\alpha_k)]_q\Big\{\Big\llbracket\Big\llbracket\llbracket E_d,E_i\rrbracket,\llbracket E_d,E_j\rrbracket\Big\rrbracket,E_k\Big\rrbracket \\
&\qquad -q^{(\alpha_d|\alpha_k)}E_i\llbracket E_d,E_j\rrbracket\llbracket E_d,E_k\rrbracket+q^{(\alpha_d|\alpha_k-\alpha_i)}\llbracket E_d,E_j\rrbracket E_i\llbracket E_d,E_k\rrbracket \\
&\qquad -q^{-(\alpha_d|\alpha_k)}\llbracket E_d,E_k\rrbracket E_i\llbracket E_d,E_j\rrbracket+q^{(\alpha_d|\alpha_j)}\llbracket E_d,E_k\rrbracket\llbracket E_d,E_j\rrbracket E_i\Big\}\sigma_dK_d\otimes E_d \\
&=(q-q^{-1})[(\alpha_d|\alpha_j)]_q[(\alpha_d|\alpha_k)]_q\Big\{\Big\llbracket\llbracket E_d,E_i\rrbracket,\Big\llbracket\llbracket E_d,E_j\rrbracket,E_k\Big\rrbracket\Big\rrbracket+q^{-(\alpha_d|\alpha_k)}\llbracket E_d,E_k\rrbracket E_i\llbracket E_d,E_j\rrbracket \\
&\qquad -q^{(\alpha_d|\alpha_j)}E_i\llbracket E_d,E_k\rrbracket\llbracket E_d,E_j\rrbracket+q^{(\alpha_d|\alpha_k)}\llbracket E_d,E_j\rrbracket\llbracket E_d,E_k\rrbracket E_i-q^{(\alpha_d|\alpha_k-\alpha_1)}\llbracket E_d,E_j\rrbracket E_i\llbracket E_d,E_k\rrbracket \\
&\qquad -q^{(\alpha_d|\alpha_k)}E_i\llbracket E_d,E_j\rrbracket\llbracket E_d,E_k\rrbracket+q^{(\alpha_d|\alpha_k-\alpha_i)}\llbracket E_d,E_j\rrbracket E_i\llbracket E_d,E_k\rrbracket \\
&\qquad -q^{-(\alpha_d|\alpha_k)}\llbracket E_d,E_k\rrbracket E_i\llbracket E_d,E_j\rrbracket+q^{(\alpha_d|\alpha_j)}\llbracket E_d,E_k\rrbracket\llbracket E_d,E_j\rrbracket E_i\Big\}\sigma_dK_d\otimes E_d \\
&=(q-q^{-1})[(\alpha_d|\alpha_j)]_q[(\alpha_d|\alpha_k)]_q\Big\{\Big\llbracket\llbracket E_d,E_i\rrbracket,\Big\llbracket\llbracket E_d,E_j\rrbracket,E_k\Big\rrbracket\Big\rrbracket-q^{(\alpha_d|\alpha_j)}E_i\llbracket E_d,E_k\rrbracket\llbracket E_d,E_j\rrbracket \\
&\quad +q^{(\alpha_d|\alpha_k)}\llbracket E_d,E_j\rrbracket\llbracket E_d,E_k\rrbracket E_i-q^{(\alpha_d|\alpha_k)}E_i\llbracket E_d,E_j\rrbracket\llbracket E_d,E_k\rrbracket+q^{(\alpha_d|\alpha_j)}\llbracket E_d,E_k\rrbracket\llbracket E_d,E_j\rrbracket E_i\Big\}\sigma_dK_d\otimes E_d.
\end{aligned}
$$

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